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Mathematics. — *Damped Vibrations of a String in a Homogeneous Field of Force.* By Prof. W. A. VERSLUYS. (Communicated by Prof. R. WEITZENBÖCK).

(Communicated at the meeting of March 31, 1928)

§ 1. Let the length of the string be l; we choose the centre of the string as origin, the x-axis in the direction of the string and the plane of vibration of the string as $x-y$ plane. The function y of x and t which represents the motion of the string, must satisfy the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} + 2w\frac{\partial y}{\partial t} - a^2\frac{\partial^2 y}{\partial x^2} = Y(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for $t > 0$ and $-\frac{l}{2} < x < \frac{l}{2}$.

This function must besides satisfy the boundary conditions: 1. $y = 0$ for $x = \pm\frac{l}{2}$ and $t \geqq 0$, 2. $y = \frac{\partial y}{\partial t} = 0$ for $t = 0$ and $-\frac{l}{2} < x < \frac{l}{2}$. Finally there must not be any discontinuity at the boundary.

§ 2. We shall now multiply the second member of (1) by the function

$$\chi(x) = \frac{4}{\pi}\sum_n \frac{(-1)^{\frac{n+3}{2}}}{n}\cos\frac{n\pi x}{l}.$$

The symbol $\sum_n$ indicates here and will always indicate further on a summation of terms for which the parameter n takes as value the odd numbers from 1 to ∞.

For $-\frac{l}{2} < x < \frac{l}{2}$ we have $\chi(x) = 1$ so that for values of x for which the equation (1) has a physical meaning, nothing in the equation is changed.

The equation altered in this way is satisfied by

$$y = \sum_n \varphi_n \cos\frac{n\pi x}{l},$$

if φ_n is a solution of the equation

$$\frac{d^2\varphi_n}{dt^2} + 2w\frac{d\varphi_n}{dt} + k_n^2\varphi_n = \frac{4}{n\pi}(-1)^{\frac{n+3}{2}} Y(t),^{1)} \quad . \quad . \quad . \quad (2)$$

where $k_n = \frac{n\pi a}{l}$.

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[1] Lord RAYLEIGH, Theory of Sound, 2nd Ed. 1894 pag. 130.

For the solution of (1) that satisfies the boundary conditions, we may write

$$y = \frac{4}{\pi} \sum_n \frac{(-1)^{\frac{n+3}{2}}}{n\sqrt{k_n^2-w^2}} \cos\frac{n\pi x}{l} \int_0^t dt'\, e^{-w(t-t')}\, Y(t') \sin(t-t')\sqrt{k_n^2-w^2}\,\text{[1]} \qquad (3)$$

This solution holds good when w is not equal to one of the numbers k_n; if $w = k_n$ the term of the sum in which this k_n occurs, must be replaced by

$$\frac{4}{n\pi}(-1)^{\frac{n+3}{2}} \cos\frac{n\pi x}{l} \int_0^t e^{-w(t-t')}\, Y(t')\,(t-t')\,dt'.$$

For t constant the series (3) in x is uniformly convergent for any closed interval of x, hence also for the interval $-\frac{l}{2} \leqslant x \leqslant \frac{l}{2}$.

For x constant the series (3) in t is uniformly convergent in any closed interval of t, hence also in the closed interval from 0 to t.

Accordingly the function defined by (3) has no discontinuity at the boundary.

If $Y(t)$ is discontinuous for the value $t = t_1$, the second differential quotiënt of y in (3) with respect to t is for any x, discontinuous for $t = t_1$ and on the lines $a(t - t_1) \pm x = \frac{ml}{2}$ (m odd).

§ 3. From equation (3) there follows for the motion of the centre ($x = 0$) when the force $Y(t')$ is constant:

$$y = \frac{4Y}{\pi} \sum_n \frac{(-1)^{\frac{n+3}{2}}}{n\sqrt{k_n^2-w^2}} \int_0^t e^{-w(t-t')} \sin\sqrt{k_n^2-w^2}\,(t-t')\,dt'$$

$$= \frac{4Y}{\pi} \sum_n \frac{(-1)^{\frac{n+3}{2}}}{n\sqrt{k_n^2-w^2}} \int_0^t e^{-wt''} \sin\sqrt{k_n^2-w^2}\, t''\, dt'', \quad . \quad (4)\ ,\ t'' = t - t'$$

$$= \frac{4Y}{\pi} \sum_n \frac{(-1)^{\frac{n+3}{2}}}{n\sqrt{k_n^2-w^2}} \left\{ \frac{\sqrt{k_n^2-w^2}}{k_n^2} - \frac{e^{-wt}\left(w \sin\sqrt{k_n^2-w^2}\,t + \sqrt{k_n^2-w^2}\cos\sqrt{k_n^2-w^2}\,t\right)}{k_n^2} \right\}.$$

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[1] Lord RAYLEIGH, loc. cit. pag. 131.

If we put $\frac{l}{2a} = c$, hence $k_n c = \frac{n\pi}{2}$, we have:

$$y = \frac{Yc^2}{2} - \frac{2Y}{c} \sum_n \frac{(-1)^{\frac{n+3}{2}}}{k_n^3 \sqrt{k_n^2 - w^2}} e^{-wt} \{ w \sin \sqrt{k_n^2 - w^2}\, t + \\ + \sqrt{k_n^2 - w^2} \cos \sqrt{k_n^2 - w^2}\, t \},$$

$$\left.\begin{aligned} &= a \left[1 - \frac{32}{\pi^3} \sum_n \frac{(-1)^{\frac{n+3}{2}}}{n^3 \sqrt{k_n^2 - w^2}} e^{-wt} \{ w \sin \sqrt{k_n^2 - w^2}\, t + \right. \\ &\qquad \left. + \sqrt{k_n^2 - w^2} \cos \sqrt{k_n^2 - w^2}\, t \} \right], \end{aligned}\right\} \quad . \quad (5)$$

where $\frac{Yc^2}{2} = a$. The line $y = a$ is the asymptote of the curve that is represented by the equation (5).

If we call the tension of the string P and the component in the direction of the y axis of the external force that acts on an element ds of the string $K\,dx$, we have the proportionality

$$P : K = \alpha^2 : Y,$$

whence

$$Y = \frac{K\alpha^2}{P}$$

and

$$a = \frac{Kl^2}{8P}.$$

The final deviation a of the centre of the string is therefore proportional to the force, the square of the length and inversely proportional to the tension.

§ 4. For the equation (4) we can write

$$y = \frac{4Y}{\pi} \int_0^t dt'' \sum_n \frac{(-1)^{\frac{n+3}{2}}}{n \sqrt{k_n^2 - w^2}} e^{-w t''} \sin \sqrt{k_n^2 - w^2}\, t'',$$

as the series under the integral sign is uniformly convergent for the interval from 0 to t.

Hence:

$$y = \frac{4Y}{\pi} \int_0^t dt'' e^{-w t''} \sum_n \frac{(-1)^{\frac{n+3}{2}}}{n} \int_0^{t''} \cos \sqrt{k_n^2 - w^2}\, t''' dt''',$$

$$= \frac{4Y}{\pi} \int_0^t dt'' e^{-w t''} \int_0^{t''} dt''' \sum_n \frac{(-1)^{\frac{n+3}{2}}}{n} \cos \sqrt{k_n^2 - w^2}\, t''', \quad . \quad . \quad (6)$$

where the order of the integration and the summation may be interchanged, as for the series we can write the sum of the two series:

$$\sum_n \frac{(-1)^{\frac{n+3}{2}}}{n} \{ \cos \sqrt{k_n^2 - w^2}\, t''' - \cos k_n t''' \}$$

and

$$\sum_n \frac{(-1)^{\frac{n+3}{2}}}{n} \cos k_n t'''.$$

The former of these two series is uniformly convergent for the interval from 0 to t and in the latter, although not uniformly convergent for any interval, yet the interchanging of the order of integration and summation is allowed.

If we differentiate (6) twice with respect to t we find:

$$\frac{d^2y}{dt^2} = \frac{4Y}{\pi} e^{-wt} \Bigg\{ - w \int_0^t dt''' \sum_n \frac{(-1)^{\frac{n+3}{2}}}{n} \cos \sqrt{k_n^2 - w^2}\, t''' +$$

$$+ \sum_n \frac{(-1)^{\frac{n+3}{2}}}{n} [\cos \sqrt{k_n^2 - w^2}\, t - \cos k_n t] + \sum_n \frac{(-1)^{\frac{n+3}{2}}}{n} \cos k_n t \Bigg\}.$$

The first two terms of this sum are continuous and the last term takes the value Ye^{-wc} for $t = c_{-0}$ and the value $- Ye^{-wc}$ for $t = c_{+0}$. Accordingly $\frac{d^2y}{dt^2}$ has a discontinuity for $t = c$; when t passes through c, $\frac{d^2y}{dt^2}$ decreases by $2\, Ye^{-cw}$.

§ 5. The expansion of $\cos \sqrt{k_n^2 - w^2}$ by TAYLOR's theorem gives

$$\cos \sqrt{k_n^2 - w^2} = \cos k_n - \frac{w^2}{1!} \frac{d \cos k_n}{dk_n^2} + \frac{w^4}{2!} \frac{d^2 \cos k_n}{(dk_n^2)^2} - \frac{w^6}{3!} \frac{d^3 \cos k_n}{(dk_n^2)^3} + \ldots,$$

as we can easily prove, that the complementary term:

$$\pm \frac{w^{2m}}{m!} \frac{d^m \cos \sqrt{k_n^2 - \vartheta w^2}}{(dk_n^2)^m}$$

approaches to zero.

If in the series we replace w by wt and k_n by $k_n t$, it becomes

$$\cos \sqrt{k_n^2 - w^2}\, t = \cos k_n t - \frac{w^2 t^2}{1!} \frac{d}{dt^2} \frac{\cos k_n t}{k_n^2} + \frac{w^4 t^4}{2!} \frac{d^2}{(dt^2)^2} \frac{\cos k_n t}{k_n^4} - \ldots.$$

$$= \Bigg\{ 1 + \frac{w^2 t^2}{1!} \frac{d}{dt^2} \int_c^t \int_0^t dt^2 + \frac{w^4 t^4}{2!} \frac{d^2}{(dt^2)^2} \int_c^t \int_0^t \int_c^t \int_0^t dt^4 + \ldots \Bigg\} \cos k_n t$$

$$= V \cos k_n t.$$

The operation V through which $\cos k_n t$ is transformed into $\cos \sqrt{k_n^2 - w^2 t}$ is independent of k_n and consists in our taking the limit of a sum the $(p+1)^{\text{th}}$ term of which is found by $2p$ times integrating $\cos k_n t$ with respect to t, alternatively between the limits 0 and t and c and t, and by differentiating the result p times with respect to t^2 and by multiplying it after that by $\dfrac{(w^2 t^2)^p}{p!}$.

For the equation (6) we now can write:

$$y = \frac{4Y}{\pi} \int_0^t dt'' e^{-w t''} \int_0^{t''} dt''' \sum_n \frac{(-1)^{\frac{n+3}{2}}}{n} V \cos k_n t'''$$

$$= Y \int_0^t dt'' e^{-w t''} \int_0^{t''} dt''' V \frac{4}{\pi} \sum_n \frac{(-1)^{\frac{n+3}{2}}}{n} \cos k_n t''',$$

as we can prove that here the operations V and $\sum\limits_n$ may be interchanged.

If $0 < t < c$ we have $t'' < c$ and also $t''' < c$, hence the sum under the operation sign $V = +1$ and

$$V \frac{4}{\pi} \sum_n \frac{(-1)^{\frac{n+3}{2}}}{n} \cos k_n t''' = \cos h\, w\, t'''.$$

Accordingly for $0 < t < c$:

$$y = Y \int_0^t dt' e^{-w t''} \int_0^{t''} \cos h\, w t''' \, dt'''$$

$$= \frac{Y}{4 w^2} \left(e^{-2wt} - 1 + 2wt \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$= a \frac{e^{-2wt} - 1 + 2wt}{2 w^2 c^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From (7) we get:

$$\frac{d^2 y}{d^2 t} = Y e^{-2wt}$$

so that the curve k, which represents the motion of the centre, is concave upward for $0 < t \leqslant c$.

Hence for $t = c_{+0}$

$$\frac{d^2 y}{dt^2} = Y e^{-2cw} - 2\, Y e^{-cw} = Y e^{-2cw} (1 - 2\, e^{cw}) < 0\,;$$

(cf. the conclusion of § 4) and the curve k is, therefore, concave downward after $t = c$. Accordingly the curve k has an apparent point of inflexion with abscissa c and ordinate:

$$b = a \frac{e^{-2cw} - 1 + 2\, cw}{2\, c^2 w^2}.$$

§ 6. In EINTHOVEN's paper on his string-galvanometer (Ann. der Phys. Bd. 21 p. 665, 1906) we find three analyses of photograms of the motion of the centre of the string in the case that there passes a constant current through the string, hence $Y=\text{const}$. The curves that represent the motion of the centre in the y-t-plane, touch the t-axis at the origin, have an asymptote $y=a_1$ and lie entirely in the strip of the plane between the asymptote and the t-axis. From the origin to a point B_1 $(t=c_1, y=b_1)$ the curves turn there concave side upward, from this point to the infinite, or practically to their coincidence with the asymptote, downward. If now the curve k is to coincide with one of the analyses of EINTHOVEN we must choose the constants in the equation for k so that 1. the asymptotes coincide, hence $a=a_1$; 2. the points of inflexion coincide, hence $c=c_1$ and $b=b_1$, from which relation follows

$$a_1 \frac{e^{-2cw}-1+2\,cw}{2\,c^2\,w^2}=b_1, \quad \ldots \ldots \ldots \quad (9)$$

so that cw may be calculated.

In the first analysis (fig. 3) of the paper (p. 669) we read $a_1=24{,}6$, $b_1=3{,}9$, $c_1=7{,}3$, whence, according to (9), $cw=5{,}76$. Table I contains a few values of y, calculated by the aid of formula (8) for the values of t on the left side column. On the right of the calculated values we find the observed values of y derived from the figure. Table II contains in the first column the values of t, in the second the values of the distances q of the points of the curve from the asymptote $(q=a-y)$ calculated by the aid of formula (5); in the third column the values of q that are derived from the figure. The greatest discrepancy between the calculated value and the value observed by EINTHOVEN is 0,83 (for $t=4c$).

TABLE I.

t	y	Observed	Difference
0.3 c	0.9223	0.72	0.2023
0.5 c	1.766	1.59	0.076
0.8 c	3.046	2.95	0.096
c	3.9	3.9	—

TABLE II.

t	q	Observed	Difference
2 c	16.74	16.28	0.46
3 c	13.44	12.72	0.72
4 c	10.803	9.97	0.833
11 c	2.341	1.7	0.641
12 c	1.881	1.35	0.531

From the second analysis (fig. 4, p. 672) it follows through measurement that $a_1=65{,}5$, $b_1=13{,}7$, and further through calculation from formula (9) $cw=4{,}22$.

The tables III and IV contain the calculated values of y and q beside the observed values.

The greatest discrepancy is 0,5 (for $t = 8\,c$).

TABLE III.

t	y	Observed	Difference
$\frac{1}{10}$ c	0.504	0.6	0.096
$\frac{1}{4}$ c	2.26	2.5	0.24
$\frac{1}{2}$ c	5.95	6.0	0.05
$\frac{3}{4}$ c	9.81	9.7	0.11
c	13.69	13.7	0.01

TABLE IV.

t	q	Observed	Difference
2 c	38.3	38.4	0.1
3 c	28.29	28.6	0.31
4 c	20.90	20.9	0
10 c	3.39	3.2	0.19
11 c	2.51	2.4	0.11

§ 7. If we choose cw so that for great values of t the calculated values of q agree with the values given by EINTHOVEN in the analyses 1 (fig. 3), if, therefore, we choose cw so that for $t = 96$ and $a_1 = 24{,}6$ formula (5) gives the measured value $q\,(96) = 1$, we find $cw = 5{,}15$.

Table V contains in the second column under q_1 the values of q calculated with these values for cw and a_1 by the aid of formula (5); in the third column we find the values derived from EINTHOVEN's analysis and in the fourth column the discrepancies between the observed and the calculated values. The maximum discrepancy is 0,4 for $t = c$.

Table VI contains for $t \leq c$ in the second column, under z_1, the values of y calculated by the aid of formula (8), in the third column the observed values and in the fourth column the discrepancies.

TABLE V.

t	q_1	Observed	Difference	$-z_2$
c	20.29	20.69	0.4	0.4152
2 c	15.94	16.28	0.34	0.3445
3 c	12.48	12.72	0.26	0.2705
9 c	2.87	2.86	0.01	0.0621
11 c	1.76	1.7	0.06	0.038
13 c	1.08	1.09	0.01	0.023

We can account for the discrepancies between the calculated and the observed values in the tables V and VI by the fact that in the calcula-

TABLE VI.

t	z_1	Observed	Difference	$-z_2$
$\frac{c}{10}$	0.179	0.12	0.159	0.1229
$\frac{c}{4}$	0.766	0.53	0.213	0.3055
$\frac{c}{2}$	1.927	1.59	0.337	0.4011
$\frac{3c}{4}$	3.119	2.72	0.399	0.4120
c	4.313	3.9	0.413	0.4131

tion we have not considered the selfinduction.[1] For a constant electromotive force the selfinduction is taken into account if in (1) we replace Y by $Y(1-e^{-vt})$, where $v=\frac{r}{L}$.

Through this the equation (4) for the motion of the centre of the string passes into

$$y=\frac{4Y}{\pi}\sum_n \frac{(-1)^{\frac{n+3}{2}}}{n\sqrt{k_n^2-w^2}}\int_0^t e^{-wt''}\{1-e^{-v(t-t'')}\}\sin\sqrt{k_n^2-w^2}\,t''\,dt''$$

or

$$y=z_1+z_2,$$

where z_1 has the same form as y in equation (4) and

$$z_2=-\frac{4Y}{\pi}\sum_n \frac{(-1)^{\frac{n+3}{2}}}{n\sqrt{k_n^2-w^2}}\,e^{-vt}\int_0^t e^{(v-w)t''}\sin\sqrt{k_n^2-w^2}\,t''\,dt'' \quad \ldots (10)$$

By applying the method followed in § 5 we find for $0<t<c$

$$z_2=-\frac{2a}{c^2}\left\{\frac{1}{2vw}-\frac{e^{-2wt}}{2w(v-2w)}+\frac{e^{-vt}}{v(v-2w)}\right\},$$

and for values of $t>c$ we can write for (10):

$$z_2=-\frac{8a}{\pi c^2}\sum_n\frac{(-1)^{\frac{n+3}{2}}}{n}\left[\frac{e^{-vt}}{(v-w)^2+k_n^2-w^2}+\right.$$
$$\left.+\frac{e^{-wt}\{(v-w)\sin\sqrt{k_n^2-w^2}\,t-\sqrt{k_n^2-w^2}\cos\sqrt{k_n^2-w^2}\,t\}}{\{(v-w)^2+k_n^2-w^2\}\sqrt{k_n^2-w^2}}\right].$$

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[1] L. S. Ornstein, These Proceedings, Vol. 17, p. 784.

A few values of $-z_2$ calculated by the aid of the latter two formulas, have been collected in column 5 of the tables V and VI. The discrepancies in column 4 of these tables appear to agree very well with the values of $-z_2$ in column 5 so that by taking the selfinduction into account the discrepancies between the observed and the calculated values become less than the errors of observation.

Psychology. — *On the Contractility of the Atrio-ventricular Bundle.* By Prof. E. D. WIERSMA.

(Communicated at the meeting of April 28, 1928).

It is well-known that simultaneous psychical contents inhibit one another. About 25 years ago HEYMANS established this for sensations, and gave a detailed description of the principles underlying these inhibition processes. These laws of inhibition hold good not only for sensations. They apply to all simultaneous psychical contents. Their mutual inhibition increases according as the qualitative and local differences between the stimuli are smaller. A weak light-sensation e.g. will be inhibited more intensely by a stronger light-sensation than by an auditory impression or an electric stimulus. The same applies to emotions, traits of character, conscious and unconscious acts of volition, reflexes etc. A few examples will serve to illustrate this.

The inborn selfish tendencies in children are less strong in adults. They are inhibited by an intensifying altruism. They are not nullified, for they return in old age, when the acquired sense of altruism weakens. Not infrequently do old men grow more selfish again and not seldom do they set greater store by the gratification of bodily wants. The vital tendencies have intensified. They can hardly resist the greater desire for smoking, eating, drinking etc. This appears most distinctly in senile dementia.

The grasping power of the right hand e.g. gets considerably weaker, when it acts simultaneously with the left hand or the legs. Writing is also impeded by other occupations engrossing the subject. Thus every voluntary movement loses in vigour, quickness, and firmness through other manifestations of the will. Flexion of the fore-arm is responded by relaxation of the tensors. The intensity of this relaxation depends on the energy of the movement, and thus the process of inhibition brings about an harmonic co-operation of the antagonistic muscles. The same is revealed in the ocular movements. The contraction of the m. abducens is responded by a relaxation of the m. internus.

Corresponding inhibitory actions impart the non-conscious motor impulses as reflexes to each other. This is proved by the following instances. The mechanism of micturition is controlled by two muscles, viz. the sphincter and the detrusor. The sphincter is generally closed, but relaxes as soon as the detrusor contracts. Many congenital reflexes disappear or become weaker through the development of higher motor functions. The sucking reflex of the neonatus disappears, when the muscles of mouth, tongue, and lips are used for other purposes, but it returns with Dementia Senilis, in which the mental powers are degenerated, and consequently can no longer exert an inhibitory action. It is quite the same with the walking-reflex and the *clinging* reflex, which belong to the first months, which disappear afterwards in consequence of the inhibition of higher manifestations of the will, but return unweakened when, in consequence of some morbid process, the inhibitory activity of the higher manifestations of the will, is lost. BABINSKI's reflex, which was superseded by reflectory flexion of the toes, returns unweakened, when in consequence of lesions of the pyramidal tracts or of deep relapses of consciousness this inhibitory influence is suspended. The patellar reflexes, whose intensity diminishes through simultaneous impulses, that proceed continuously towards the periphery to facilitate first voluntarily, but afterwards automatically or reflectorily the easiest position of the extremities, get higher again, as soon as the centrifugal impulses can no longer act correctively on account of lesions of the spinal cord or the brain.

This inhibition-law is of a generally biological significance. It manifests itself in normal and abnormal spiritual life, in the physiological and pathological motor processes, in thinking and in feeling, and with all simultaneous centripetal and centrifugal impulses, among which also the growth of the limbs can be classed. Of the latter PAUL GODIN's [1]) inquiry furnishes a striking example. He compared every half year in children of from 13½ to 17½ years the increase in length of the upper leg with that of the lower leg, the growth of the upper arm with that of the lower arm, and also the growth of the length of the lower leg with the increase of its girth. The remarkable thing in this research is, that the growth of each part of the leg is rhythmic in that the intervals of stagnation in the growth of the upper leg coincide with a stronger growth of the lower leg, and conversely. Similarly the tardier growth of the upper arm coincides with a stronger growth of the lower arm and vice versa. Furthermore he could establish that the increase of the length of the leg inhibits the increase of girth, and the reverse. The increase in thickness alternates with growth in length.

It would not be difficult to enlarge the number of these examples. They imply that, when for a given process more energy is required, it is withdrawn from another, that differs little from it qualitatively and often also locally. The question naturally rises, whether or not other inhibition-processes, such as the influence of the vagus-stimulus on the cardiac action, should be looked at from this point of view. After the discovery of the phenomenon by the WEBER brothers in 1845 no satisfactory explanation has been found for it. The supposition that also here two synchronous, competing motor impulses are the causative factors, is supported by the investigations of FANO, ROSENZWEIG, BOTTAZI and GASKELL, who found in the auricle of the tortoise-heart two layers of muscles, the inner one being innervated by the vagus, the outer one by the sympathicus. Greater activity of the inner layer incited by vagus-stimulation, inhibited the contractility of the outer one, and conversely greater activity of the outer layer, incited by sympathicus-stimulation, induced a lessened activity of the inner one.

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[1]) PAUL GODIN, Recherches anthropométriques sur la croissance etc. 1903.

It is notorious that also in the mammalian heart there are two layers of muscles, viz. the ordinary heart-muscle and the atrioventricular bundle, situated just below the endocard. This layer was discovered by HIS in 1895. Now it imports to us to know in the first place whether this bundle is capable of contraction, and secondly whether vagus-stimulation can have an action upon that contractility. For the present I am positive only about the first point.

The bundle of HIS connects the auricles to the ventricles of the heart. It serves to convey stimuli ; whether it had a contractility of its own was still an open question. ERLANGER 1), who studied in 1912 the conduction of the bundle, says that on stimulating the bundle he invariably witnessed a systole. Once or twice he fancied to have detected a motion of the bundle. "In only one case did the false tendon seem to move independently of the heartwall". In 1921 and 1922 an investigation was started in the laboratory of the Groningen University 2) concerning the contractility of the bundle. When after some observations it appeared that the spontaneous contractions could be observed with the unaided eye, I endeavoured to register these movements. I deemed it interesting, because then it would be easier to intercompare the movements of the bundle under various conditions. Researches made in 1927, in which the influence of different stimuli was ascertained, substantiated previous results and yielded some new ones. I shall revert to them later on. In 1922 ISHIHARA also observed the contractions of the bundle in the hearts of rabbits, cats and dogs, perfused with LOCKE's solution. He could watch those contractions, which were synchronous with the cardiac movement, with a pocket-lens. In 1924 PICK's 3) investigations appeared. He had studied under the microscope the movement of the isolated fibre of PURKINJE in dogs, cats, and rabbits, in oxygenated, heated RINGER-solution. He noted an increase of the frequency and the intensity of the contractions under the influence of strophantin in $^1/_{1000000}$ dilution. The frequency rose from 24 to 88, and at length to 120, but at this quick rate the movements became smaller and less pronounced. Vagus-stimulating substances, such as acetylcholin, and pharmaca that paralyzed the vagus, such as nicotin and atropin, did not interfere with the contractions at all. From this PICK infers that the atrioventricular system does not contain vagal nerve-elements. PICK still

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1) ERLANGER, American Journ. of physiol. 1912.

2) E. D. WIERSMA, These Proceedings **30**.

E. D. WIERSMA, Archives néerlandaises de physiologie de l'homme et des animaux. Tome 7 1922.

3) E. P. PICK, Ueber das primum und ultimum moriens im Herzen. Klin. Wochenschrift 1924.

calls attention to other important factors. Some parts of the heart are highly resistant, others only very little. The site of the greatest resistance, i.e. the place where pulsations persist longest, or from which movements first reappear under favourable conditions after apparent death, is called *ultimum moriens*. Besides this PICK assumes a *primum moriens*, i.e. the part that dies off first. The ultimum moriens is the bundle of HIS, the primum moriens is situated where that bundle is connected with the heartmuscle. Whereas the supply of oxygen does not exert any influence on the ultimum moriens, the primum moriens is very sensitive to it. A disturbed contact between the atrioventricular system and the heartmuscle can be restored by supply of oxygen. In 1925 the contractions of the bundle were also noted by LAPIQUE and VEIL.

My curves were registered in the following way: Directly when the animals had been killed, the sheep's hearts were conveyed in their own blood from the slaughter-house to the laboratory, a distance of 10 min. The heart was then opened immediately. In the right ventricle the bundle is isolated most, and is most clearly visible. On opening the heart very often rhythmic contractions are distinctly to be seen with the naked eye. They are often regular, and their frequencies vary from 20 to 60 per minute. Sometimes the contractions are irregular and the frequency is sometimes lower than 20. The contractions start in the auricle and proceed sinuously to the heart-muscles, not infrequently can they be seen to pass on to the heart-muscle. It gives one an impression, as if the conduction is effected by the contractions. After a quarter of an hour or often earlier the contractions disappear. For the registration of the curves the removed part of the bundle is suspended in a heated RINGER or LOCKE-solution, which is continuously perfused with oxygen. The spontaneous contractions of the bundle are mostly so distinct that a cinematographic photo can be taken of them. In almost all cases the registration of the movement succeeds. The curves show deflections of various magnitude and frequency. With faradic stimulation the deflection and the frequency increase. There is still an after-effect of the stimulus for a brief-period. In other cases the contractions cease, when the stimulus is discontinued.

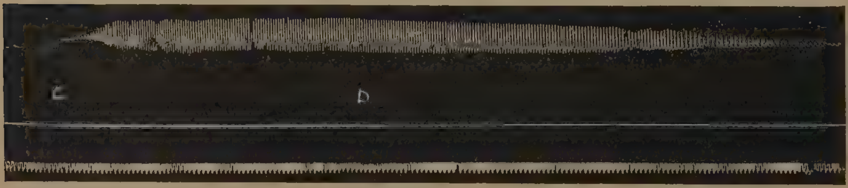

Fig. 1.[1])

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[1]) This curve is to be read from left to right, all the others from right to left.

Not infrequently the contractions are irregular.

It looks as if a tonic contraction results from the faradic stimulus, to which rapid contractions of different magnitude are superposed Sometimes they resemble fibrillar contractions (Fig. 2).

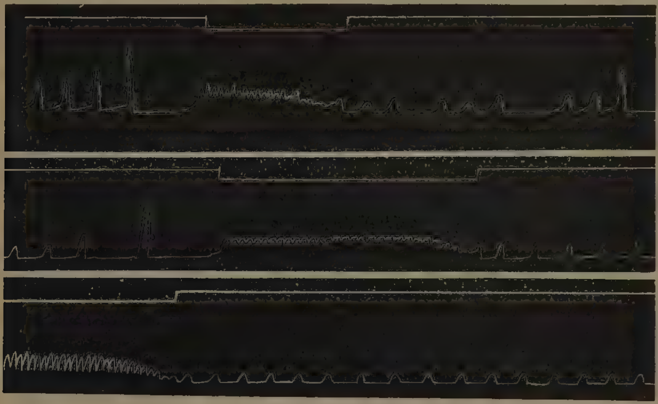

Fig. 2.

In these experiments the influence of pharmaca could easily be illustrated by curves. The action of a $^{1}/_{1000000}$ dilution of strophanthin had a very great effect upon the frequency and the magnitude of the deflection. Fig. 3.

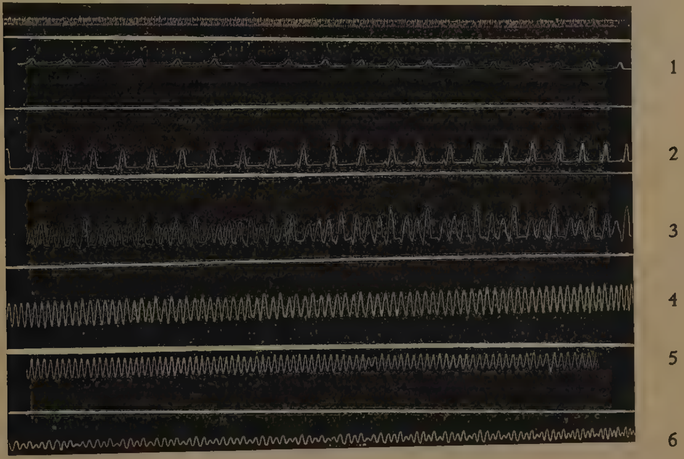

Fig. 3.

1. Spontaneous contractions.

Strophanthin action:

2. After 4 minutes. 3. After 8 minutes. 4. After 12 minutes.
5. After 16 minutes. 6. After 20 minutes.

This tallies with PICK's experience. It also appears from the curves that with a very high frequency the intensity of the contractions diminishes.

Nicotin appears to have no influence upon the contractions. (Fig. 4).

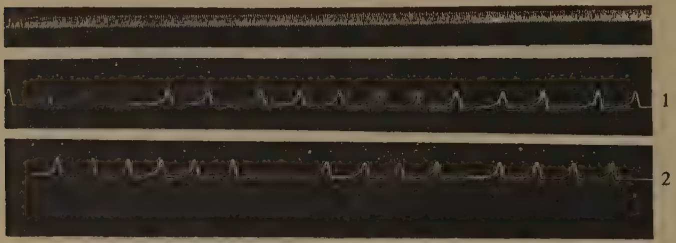

Fig. 4.

1. Normal.
2. Nicotin-action.

The influence of atropin could not yet be excluded with certainty, for Fig. 5 shows that in curve N^0. 2, the deflection is a little less high on account of the atropin, and that, in curve N^0. 3, the excursion becomes much higher after the addition of strophanthin.

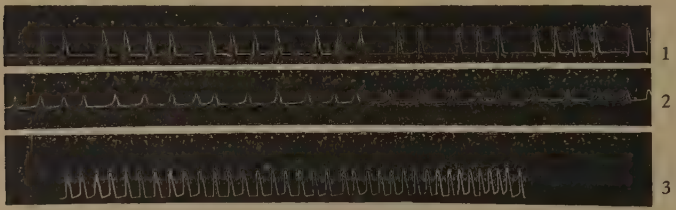

Fig. 5.

1. Normal.
2. Atropin-action.
3. Addition of strophanthin.

Neither did the results always agree with regard to the action of pharmaca that stimulate the vagus, such as acetyl-cholin.

In one curve the deflections, obtained by a faradic stimulus were not, or only little, modified by acetylcholin, but in another curve this is really the case. A dilution of $^1/_{1000000}$ gave after a few seconds (Fig. 6) (curves 4, 5 and 6) an indubitably larger deflection. Subsequently atropin, a pharmakon, that paralyzes the vagus, was added in a dilution of $^1/_{25000}$. Already after some seconds (curve 7) the deflection lessened, and ultimately one single feeble deflection is still observed, and finally also this one disappears. When after this the bundle was suspended in another LOCKE-solution without any pharmakon (curve 8) the paralyzing effect of atropin disappeared, upon which the bundle resumed its contractions, as before.

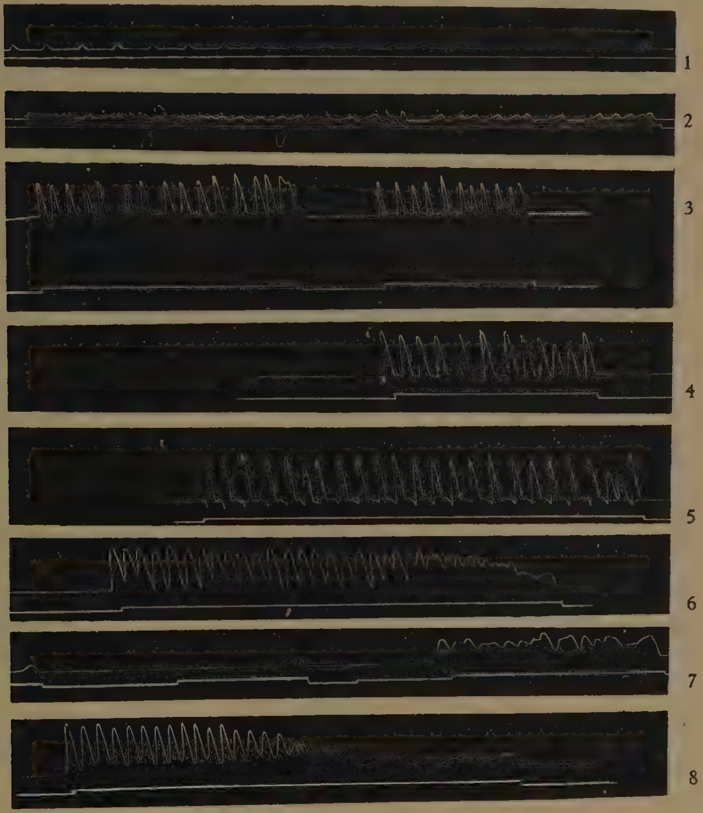

Fig. 6.

1. Normal.
2. Faradic stimulus.
3. Stronger faradic stimulus.
4. Acetylcholin (immediately).
5. Acetylcholin (after 15 minutes).
6. Acetylcholin (after 30 minutes).
7. After addition of atropin.
8. After renewing the liquid.

So these experiments tend to show that it is necessary to repeat once more the action of acetylcholin and of atropin.

I still wish to point to the influence of the temperature (Fig. 7). At 37° C. the frequency is rather high. It falls considerably at 42° C., but the intensity of the deflection remains the same and the regularity of the rhythm persists, so that there is no question about exhaustion. Not before the temperature rises to 46° C. the frequency rises considerably, the

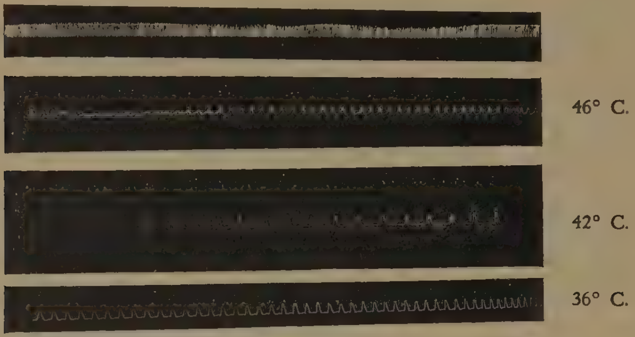

Fig. 7.

deflection diminishes and the rhythm becomes irregular. In many experiments it appeared that also at lower temperatures, even at 23° C. contractions were still clearly visible, though they were less strong than at 37° C.

It appears from the above that of the two questions I propounded, the first must be answered in the affirmative, i.e. there exists distinct contractility of the bundle of HIS. An answer to the second question can be given only when more curves are constructed.

Physics. — *Measurements on the disturbance of the supra-conductivity of thallium by magnetic fields.* Communication N⁰. 191*b* from the Physical Laboratory at Leyden. By W. TUYN. (Communicated by Prof. W. H. KEESOM).

(Communicated at the meeting of May 26, 1928).

§ 1. Measurements on the dependence of the magnetic threshold value on temperature pointed in the cases of tin and indium to the relation [1]:

$$H_{(1/2)} = h(T_S^2 - T^2);$$

in this $H_{(1/2)}$ is the magnetic field, which restores half of the resistance, h a constant and T_S the vanishing point temperature. These measurements were made on micro-crystalline wires, in fields, which were gradually increased in strength at constant temperature; the direction of the field was transverse to that of the current, so the transverse effect has been determined [2].

The measurements with resistances of lead [3] did not contradict this relation, but were too few in number for a conclusion to be reached. Lead is the most important metal for an investigation into the dependence of the magnetic threshold value on temperature: this can be measured over a region of more than 6 degrees. Difficulties are the use of the helium vapour cryostat and the application of strong fields; owing to pressure of other work, it has been impossible to make the plotted measurements with the aid of a coil, newly constructed for this purpose.

However, in the autumn of 1924 measurements of the same kind were made with a resistance of thallium, but between 2°.5 and 1°.6 K. only. Though open to extension, the results obtained may be quoted as follows.

We used *Tl*—1916 -*VIII*, with which resistance the supra-conductivity of thallium was measured for the first time [4]; the method used was rhythm persists, so that there is no question about exhaustion. Not before the temperature rises to 40° C. the frequency rises considerably, the

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[1] W. TUYN and H. KAMERLINGH ONNES, Journ. Franklin Institute 201, 379, 1926, Comm. Leiden N⁰. 174*a*.

[2] W. J. DE HAAS, G. J. SIZOO and H. KAMERLINGH ONNES, These Proceedings 29, 947, Comm. Leiden N⁰. 180*d*, p. 69, found this formula only approximately true for the *longitudinal* effect with mercury. As they noticed, $H_{(1/2)}$ has a physical meaning only in the case of measurements made with micro-crystalline wires, as were the tin and indium-resistances used, whilst a comparison with the measurements cited must be restricted to the ascending line of the hysteresis figure.

[3] W. TUYN and H. KAMERLINGH ONNES, l.c., p. 397, Comm. Leiden N⁰. 174*a*, p. 25.

[4] H. KAMERLINGH ONNES and W. TUYN, These Proceedings 25, 443, Comm. Leiden N⁰. 160*a*.

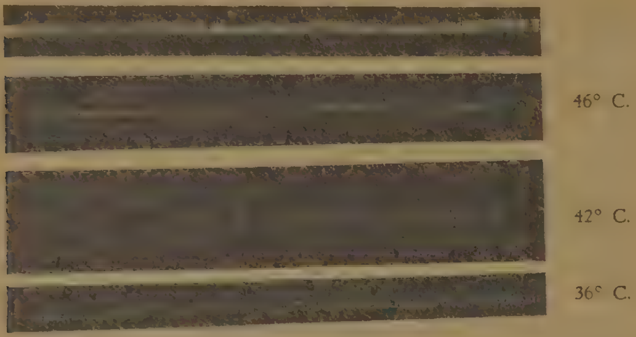

Fig. 7.

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We used *Tl*—1916—*VIII*, with which resistance the supra-conductivity of thallium was measured for the first time [4]); the method used was that of compensating the potential at the extremities of the known and of an unknown resistance, connected in series, by the aid of a DIESSELHORST compensation apparatus, free from thermal forces, supplied by O. WOLFF. From the potential differences we calculated resistances as if OHM's law were valid also below the vanishing point temperature. As with former measurements fields were obtained with the aid of coils; sometimes these were placed round the cryostat and sometimes immersed in the liquid helium. Their particularities are omitted; only the quantities *A* are given in the tables I—V, as a measure for the homogeniety of the field for the part of the axis of the coil[1], taken up by *Tl*—1916—*VIII*. The resistance was placed in a cryostat provided with a stirring apparatus; vapour pressures of the heliumbath were read off on a mercury manometer with a cathetometer; they gave the temperatures of the bath[2]. For further experimental details we refer to former papers.

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[1]) W. TUYN and H. KAMERLINGH ONNES, Journ. Franklin Institute **201**, 379, 1926, Comm. Leiden N⁰. 174*a*.

[2]) W. J. DE HAAS, G. J. SIZOO and H. KAMERLINGH ONNES, These Proceedings **29**, 947, Comm. Leiden N⁰. 180*d*, p. 69, found this formula only approximately true for the *longitudinal* effect with mercury. As they noticed, $H_{(^1/_2)}$ has a physical meaning only in the case of measurements made with micro-crystalline wires, as were the tin- and indium-resistances used, whilst a comparison with the measurements cited must be restricted to the ascending line of the hysteresis figure.

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§ 2. The results of the measurements for the *ascending* line follow in tables I—V; some points determined on the descending line, indicate that thallium also shows the hysteresis phenomenon[3]. The vertical component of the terrestrial magnetic field is taken into account, when fixing the mean field intensity.

An examination, in the manner previously described[4], whether inhomogeniety can wholly explain the difference between initial and final field, shows that this is not the case. DE HAAS, SIZOO and KAMERLINGH ONNES explain the rest of this difference by the hypothesis that the field, which brings back resistance in a supra-conductive crystal, depends on the angle between crystal axis and direction of the field. By assuming further that this dependence is a function of temperature, one might explain in this way the change in inclination of the transition curve with temperature[5]; this change was found also with the measurements of *Tl*—1916—*VIII*.

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[1] The intensity of the field for a point on the axis with coordinate z, is:

$$H_z = 4\pi ni \, . \, {}^1/_2 \left\{ \frac{z}{\sqrt{R^2+z^2}} - \frac{z-l}{\sqrt{R^2+(z-l)^2}} \right\} = 4\pi ni \, . \, A_z \, ;$$

0 and l are the coordinates of the ends of the coil. The resistance being placed as symmetrically as possible in the coil, A is maximum in the middle, minimum at the ends of the part taken in by the resistance. The mean field intensity, that is the mean of all the intensities over the same part of the axis, can easily be calculated from the formula by integration.

[2] After the formula, calculated by H. KAMERLINGH ONNES and SOPHUS WEBER, These Proceedings **18**, 493, Comm. Leiden N⁰. 147*b*.

[3] For this phenomenon see G. J. SIZOO, W. J. DE HAAS and H. KAMERLINGH ONNES, These Proceedings **29**, 221, Comm. Leiden N⁰. 180c.

[4] W. TUYN and H. KAMERLINGH ONNES, l.c., p. 392, Comm. Leiden N⁰. 174*a*, p. 20.

[5] As abscissa in the graphical representation, the field intensity has to be taken, expressed as a percentage of the initial or final field intensity.

TABLE I. Tl—1916—VIII, transverse.

$p_{helium} = 30.6—30.7$ int. mm Hg. $T = 2°.21_0—2°.21_6$ K.[1]

Measuring current $= 4._0$ mA.

Coil *P*.

$A_{max.} = 0.994$, $A_{min.} = 0.990$, ratio $= 0.996$.

$A_{mean} = 0.993$. $H_{mean} = 43.8_0\ i$ (i in amp.).

Mean field intensity	$R_{Tl—1916—VIII}$
35.5 gauss	0.0067_2 Ω
33.3	0.0066_1
32.8	0.0062_7
32.2	0.0047_3
32.0	0.0038_6
31.1	0.0022_6
30.2	0.0013_3
29.3	0.0008_1
28.5	0.0004_6
27.6	0.0002_9
26.7	0.0001_9
22.3	0.0000_0

TABLE II. Tl—1916—VIII, transverse.

$p_{helium} = 30.8—30.9_5$ int. mm Hg. $T = 2°.20_3—2°.20_6$ K.

Measuring current $= 4._0$ mA.

Coil *W*.

$A_{max.} = 0.783$, $A_{min.} = 0.744$, ratio $= 0.950$.

$A_{mean} = 0.769$. $H_{mean} = 35.2_7\ i$ (i in amp.)

Mean field intensity	$R_{Tl—1916—VIII}$
37.1 gauss	0.0066_8 Ω
36.4	0.0066_0
35.7	0.0064_2
35.3	0.0061_6
35.0	0.0058_1
33.9	0.0044_8
32.9	0.0028_6
32.2	0.0015_4
31.5	0.0009_8
30.1	0.0004_5
28.6	0.0001_9
27.9	0.0000_8
27.2	0.0000_6

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[1] The thousandths of a degree are shown in this table and the following ones only in order to demonstrate the constancy of the helium bath.

TABLE III. Tl—1916—VIII, transverse.

$p_{helium} = 30.2$—30.4 int. mm Hg. $T = 2^{\circ}.19_7$ K.

Measuring current $= 4._0$ mA.

Coil Q.

$A_{max.} = 0.994$, $A_{min.} = 0.990$, ratio $= 0.996$.

$A_{mean} = 0.993$. $H_{mean} = 110.5_1\ i$ (i in amp.).

Mean field intensity	$R_{Tl—1916—VIII}$
38.0 gauss	0.0067_4 Ω
36.9	0.0067_0
36.3	0.0064_9
35.8	0.0061_4
34.7	0.0035_6
33.6	0.0018_5
32.5	0.0011_8
31.4	0.0006_1
30.3	0.0003_0
29.2	0.0001_3
28.1	0.0000_6
26.9	0.0000_4
25.8	0.0000_3
24.7	0.0000_0

TABLE IV. Tl—1916—VIII, transverse.

$p_{helium} = 11.5$—11.8 int. mm Hg. $T = 1^{\circ}.84_6$—$1^{\circ}.85_5$ K.

Measuring current $= 4._0$ mA.

Coil W.

$A_{max.} = 0.783$, $A_{min.} = 0.744$, ratio $= 0.950$.

$A_{mean} = 0.769$. $H_{mean} = 35.2_7\ i$ (i in amp.).

Mean field intensity	$R_{Tl—1916—VIII}$
71.0 gauss	0.0066_5 Ω
69.6	0.0063_5
68.8	0.0059_0
67.4	0.0049_5
66.0	0.0026_8
63.9	0.0018_1
61.1	0.0006_2
56.9	0.0001_2
53.3	0.0000_6

TABLE V. Tl—1916—VIII, transverse.

$p_{helium} = 10.8_5 - 11.0_5$ int. mm Hg. $T = 1°.82_6 - 1°.83_2$ K.

Measuring current = $4._0$ mA.

Coil Q.

$A_{max.} = 0.994$, $A_{min.} = 0.990$, ratio = 0.996.

$A_{mean} = 0.993$. $H_{mean} = 110.5_1\ i$ (i in amp.).

Mean field intensity	$R_{Tl-1916-VIII}$
67.8 gauss	$0.0031_1\ \Omega$
65.6	0.0021_4
63.4	0.0013_5
61.2	0.0009_2
59.0	0.0003_5
54.6	0.0000_7
52.4	0.0000_4
50.2	0.0000_1

§ 3. We derived graphically from each table the magnetic field in which the resistance is half restored; we tried to take account of the inhomogeniety of the fields.

TABLE VI. Transverse effect.

T	$H_{(^1/_2)}$
2°.47 K.	0 gauss
2.21_5	31
2.20_5	$32._5$
2.19_5	34
1.85	65
1.83	67

With thallium the transverse effect does not satisfy a quadratic relation as closely as in the case of tin; deviations up to 5 °/₀ present themselves between the observed values and those calculated from the equation $H_{(^1/_2)} = -25.4\ T^2 + 154.9$. It must not be forgotten that the temperature scale is not yet accurately known in this region; an increase in value of the vanishing point temperature (2°.47 K.) with 0.02 to 0.03 of a degree would give a better agreement. The constant h in $H_{(^1/_2)} = h(Ts^2 - T^2)$ appears to be larger than for tin and indium.

Chemistry. — *Velocities of Nitration* [1] (Provisional Communication). By F. H. COHEN. (Communicated by Prof. A. F. HOLLEMAN).

(Communicated at the meeting of June 30, 1928).

I. *Introduction.*

Only very few kinetic measurements of nitrations have been made as yet. GIERSBACH and KESSLER [2]) and BRÄUER [3]) have studied the nitration of benzene in nitro-benzene, but they did not obtain reaction velocity constants. Besides these communications in the older literature there have, so far, only appeared publications on this subject by MARTINSEN [4]), KLEMENC [5]), and WIBAUT [6]).

MARTINSEN has measured the nitration velocity of some nitro- and dinitro-derivatives of benzene in strong sulphuric acid, and also of phenol and p-cresol in water. The values for k (in sulphuric acid), calculated according to the bimolecular reaction equation $RH + HNO_3 \rightarrow R.NO_2 + H_2O$ present a very satisfactory constancy at 25° and 46°; measured at 0°, however, they sensibly diminish in the course of the experiment, especially in nitro-benzene. MARTINSEN, therefore, considers the possibility that the reaction after all does not quite perfectly agree with the bimolecular scheme. He has further examined the influence of nitrous acid on the reaction velocity. It appeared to be equal to zero here.

The nitration of phenol and p-cresol in water has a complicated course; by-products are formed, and the experiment yields no constant values for k, *points, on the other hand, to a distinct auto-catalytic course of the reaction.* This is to be attributed to the nitrous acid that is formed in the liquid (proved colorimetrically according to ERDMANN [7]), as the addition of only slight quantities of nitrite already considerably increases the velocity of reaction. If the HNO_2 is, however, removed by the addition of *urea, no nitration* takes place. The molarity of the urea was 0.05, that of the HNO_3 and the C_6H_5OH 0.6 resp. 0.075.

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[1]) This subject was given me by Prof. WIBAUT to treat as a thesis for the doctorate; I started the measurements communicated here, when I was employed as an assistant at the physiological laboratory of Amsterdam; when this had come to an end Prof. VAN RIJNBERK gave me an opportunity to continue my investigation; I gladly express my indebtedness to him for this kindness.

[2]) Z. Phys. Ch. **2**, 676 [1888].

[3]) Dissert. Heidelberg [1899].

[4]) Z. Phys. Ch. **50**, 385 [1905]; **59**, 605 [1907].

[5]) Monatsh. f. Ch. **35**, 451 [1914]; **39**, 451 [1918]; Z. f. anorg. u. allgem. Ch. **141**, 231 [1924].

[6]) Rec. **34**, 241 [1915].

[7]) B **33**, 210 [1900].

KLEMENC has made measurements on phenols, dissolved in ether. The process of the reaction appeared, however, to be very complicated in these very readily oxidizable benzene derivatives; hence, with a few exceptions, the author has not succeeded in obtaining nitration constants that were in any way satisfactory. In his latest treatise on this subject [1]) he assumes that the nitration is due to an acid $H_2N_3O_8$; its character is, however, still pretty hypothetical, as KLEMENC has to admit himself.

The two principal facts established from this investigation are: 1. a solution of HNO_3 in ether which is perfectly free from HNO_2 or NO_2, does *not* nitrate; 2. if the nitric acid is present in excess, no nitration takes place or only a slight one.

WIBAUT has measured the velocity of nitration of benzene, toluene, chloro- and bromo-benzene in anhydrous acetic acid; for toluene at 0°, for the other three substances at 25°. With a single exception the concentration of the HNO_3 was always about 0.75, that of the benzene derivative about 0.5. The calculation of the k's was made according to the bimolecular reaction equation. As appears from the tables referring to this [2]), the values of the k's in benzene and in chloro-benzene are pretty constant; the departures from the mean are, however, sometimes very great indeed (in benzene e.g. up to 28 $^0/_0$), but this is attributed by the author to errors of observation. The means of the parallel experiments inter se are in much better agreement. In bromo-benzene the k's increase in the course of the reaction, in toluene they decrease. For benzene and chloro-benzene at 25° the following values were found: $k = 0.0025$, resp. 0.0020.

II. *The Author's Experiments.*

These form a direct continuation of the above-mentioned investigation by WIBAUT. Most experiments have been made by me in anhydrous acetic acid as a solvent; I have, however, also performed a few measurements in glacial acetic acid.

The anhydrous acetic acid which I employed, was that of the firm of DE HAEN "doppelt gereinigt". The reactions to detect Cl^- and SO_4^{--} were negative; from analyses the content appeared to be about 98 $^0/_0$. Some experiments in which I used MERCK's preparation "Zur Analyse, frei von höheren Homologen", yielded no different results; DE HAEN's preparation is, accordingly, sufficiently pure for my purpose. The benzene was a perfectly pure preparation with a melting point of 5°.48. I obtained the absolute nitric acid by distillation of pure 65 % HNO_3 with double the quantity by weight of strong sulphuric acid in vacuum in an apparatus entirely made of glass. The content was as a rule about 99.5 %. After the preparation it was immediately placed in the ice-box.

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[1]) Z. für anorg. u. allgem. Chem. **141**, 236 [1924].

[2]) Compare for this the original.

Analysis of the samples. This was made by the method also used by WIBAUT, which consists in a quantity of the reaction mixture pipetted off, flowing into an excess of potassium hydroxide of the spec. weight of 1.3; the KNO_3 formed was then determined according to *Devarda*, after removal of the nitro-benzene formed through shaking out with benzene. By means of a number of experiments with a known quantity of KNO_3 I convinced myself that the accuracy of this method is sufficient.

The rôle of the HNO_2. Considering the study of the rôle of the HNO_2 in these experiments of great importance, I have tried to determine its concentration both at the beginning and during the reaction. This was done colorimetrically according to GRIES—ILOSVAY [1]) in the samples of the reaction mixture diluted in a suitable way [2]). The nitric acid was rendered innoxious by the addition of solid Na-acetate.

Result of some experiments. Table 1 renders an experiment at 25° [3]); the determinations were all made in duplicate; urea was not added; only the initial concentration of the HNO_2 was measured. Leaving the first measurement out of account, it is seen that the k (calculated according to the bimolecular equation $k = \frac{2.3025}{t(C_0 - C'_0)} \log \frac{C'_0 C}{C_0 C'}$) remains very well constant over a conversion range of more than 50 % (probably over a still greater range: the interruption of the measurements during the night prevented me from ascertaining this). Not until the end of the reaction do the k's increase greatly, a fact which I found in almost all my experiments; though the relative errors at the beginning and at the end of the reaction are, of course, much greater than in the middle part, this cannot account for the fact that the deviations are always found only in a positive sense.

The value I find for k, is therefore somewhat lower than that found by WIBAUT (0.0025).

I will now proceed to communicate an experiment, also made at 25°, but with addition of ± 8 mg of urea. This quantity is so small that in an analytical respect, it cannot exert any disturbing influence. HNO_2-

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[1]) Cf. LUNGE—BERL, Chem.-techn. Unters. Meth. I, 853 [1921].

[2]) In this besides HNO_2 also $NO_2(N_2O_4)$ will occur, which, on being poured out into water, yields $HNO_2 + HNO_3$: hence the HNO_2-concentration found colorimetrically will be somewhat smaller than the sum of the concentrations $[HNO_2] + [NO_2]$ in the reaction liquid. This has, however, no influence on the order of magnitude. In the tables 1 and 3 it has been assumed that exclusively HNO_2 is present.

[3]) The oscillations of temperature did not amount to more than ± 0°.02.

determinations have not yet been carried out in this kind of experiments (Table 2). As is seen, the experiment again yields constants, but now the value of k is very much smaller. The conclusion is obvious that the HNO_2 considerably accelerates the reaction catalytically.

TABLE 1 (*Benzene*, temp. = 25° C.)

Initial concentrations
$HNO_3 = 0.6682$
$C_6H_6 = 0.4385$
$C_0 - C'_0 = 0.2297$
$HNO_2 = 0.0011$

No urea added

t 1)	c	k
0	0.6682	—
56.4	0.6191	(0.00329)
131.5	0.5905	236
224	0.5529	225
451	0.4762	229
574	0.4417	237
1352	0.2742	(450)
1590	0.2420	(700)

Mean 0.00232

After 574′ 51.6 % converted
Greatest deviation from the mean 3.0 %

TABLE 2 (*Benzene*, temp. = 25° C.)

Initial concentrations
$HNO_3 = 0.6808$
$C_6H_6 = 0.4808$
$C_0 - C'_0 = 0.2000$

About 8 mg. of urea added

t	c	k
0	0.6671	—
109.9	0.6230	0.00140
186.4	0.6001	131
293	0.5610	144
410	0.5282	145
546	0.5017	139
654	0.4484	(179)
1551	0.2627	(347)

Mean 0.00140

After 546′ 37.2 % converted
Greatest deviation from the mean 6.4 %

Reaction of the HNO_3 with the solvent. Up to now no attention had been given to possible action of the HNO_3 caused by the anhydrous acetic acid itself. In the literature some records are, indeed, found of this reaction, chiefly in connection with the preparation of tetranitro methane 2), but the number of equivalents of the HNO_3 is then always of the same order of magnitude as of the anhydride. WIBAUT supposed 3) that acetyl nitrate is formed in the nitration experiments described in his publication, which would then be the active agent in the nitration.

Attempts to isolate this substance from a diluted solution of HNO_3 in anhydrous acetic acid have, however, not succeeded. However this may be, it appears from Table 6 of his publication 4) that no nitration

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1) In these experiments time is always expressed in minutes.

2) Cf. e.g. PICTET and GENEQUAND, B. **36**, 2225 [1910]; CHATTAWAY, Soc. **97**, 2099 [1910].

3) Loc. cit. p. 245.

4) Loc. cit. p. 250.

takes place with excess of gramme-molecules of chloro-benzene; nor have I found nitration in a similar experiment with benzene. In these cases the concentration of the HNO_3 was, however, much smaller than usual. From this the conclusion had been drawn that the titer of the HNO_3, when there is no benzene derivative present, does not decrease. It has, however, appeared to me from experiments made on purpose to study this, that it is very certainly the case. This can, of course, not be owing to a conversion into acetyl nitrate, as this would have no influence on the titer found. In Tables 3 and 4 some experiments are recorded from which this decrease of titer appears.

It will be seen that this is very considerable. It may be read from the figure (curves II and III) that the concentration remains constant for about 180 minutes, after which it decreases *linearly with the time* during a great interval. This is particularly clearly to be seen in curve III. This need not mean that this relation is also theoretically true; the course of the curve sooner suggests the existence of two points of inflection, and it may therefore represent a consecutive reaction; practically, however, the middle portion is straight.

TABLE 3.

(Experiment without benzene, temp. = 25° C.)

No urea added

t	c_{HNO_3}	c_{HNO_2}
0	0.7142	0.00030
34.7	0.7157	
105.9	0.7137	
142.7	0.7139	
189.7	0.7015	0.0089
270	0.6754	
335	0.6462	
414	0.6181	0.0157
716	0.5121	
720	0.5056 [1]	
909	0.4485	
998	0.4216	
1083	0.3921	

TABLE 4.

(Experiment without benzene, temp. = 25° C.)

15 mg. of urea added

t	c
0	0.7221
53	0.7168
179	0.7185
376	0.7018
584	0.6339
1554	0.3338
1811	0.2901

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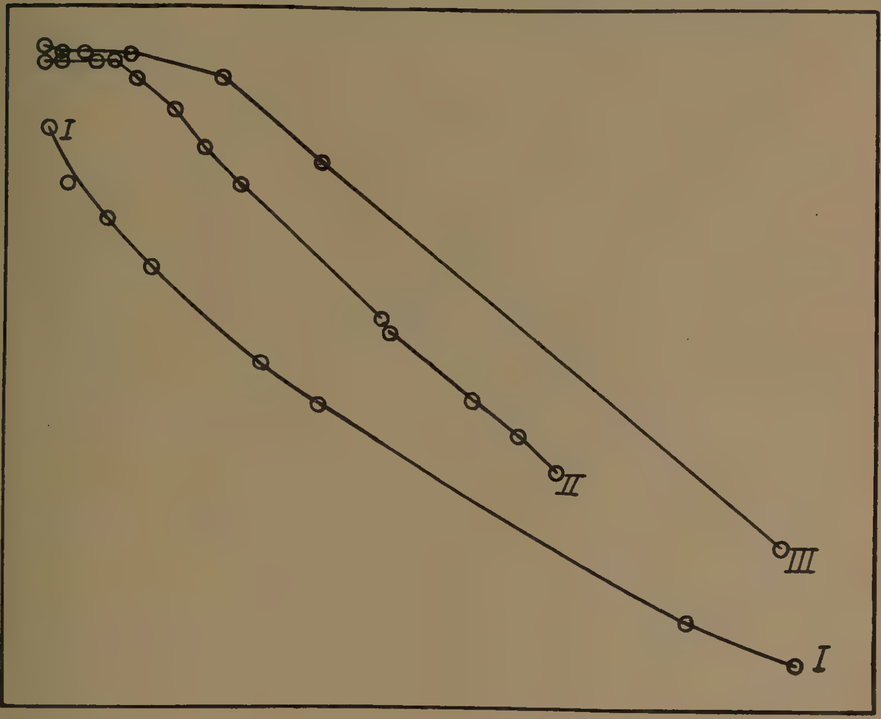
[1] New experiment, started the preceding day in the evening.

If it is now assumed that the reactions of the HNO_3, on one side with the benzene, and on the other side with the solvent, proceed



independently of each other, this gives rise to the two simultaneous differential equations

$$\frac{dx}{dt} = k(A - x - y)(B - x) \quad \ldots\ldots\ldots \quad (1)$$

$$\frac{dy}{dt} = a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

in wich A and B are the initial concentrations of nitric acid, resp. benzene.

Equation (2) integrated yields $y = at$, as for $t = 0$ y is $= 0$.

Substitution of this in (1) yields:

$$\frac{dx}{dt} = k(A - x - at)(B - x) \quad \ldots\ldots\ldots \quad (3)$$

This differential equation would then have to represent the course of the nitration, at least from about 3 hours after the beginning of the reaction. We now see from Table 3 that in the interval of 720—909 minutes the titer of the HNO_3 decreases by an amount of 0.0571 mol.; per minute this is 0.000302 mol. Table 4 gives this same amount over a still greater interval of conversion. We may therefore put $a = 0.000302$ in equation (3). If in Table 1 we now consider the interval of 224—574 min., the HNO_3 would have been diminished by $350 \times 0.000302 = -0.1057$ mol. through the reaction with the solvent.

The total decrease being 0.1112 mol., only 0.0055 mol. remains for the reaction with the benzene, hence an impossibly small amount. It is, indeed, easy to choose such a range of conversion that the decrease of the benzene concentration would change into an increase. It therefore appears (and the satisfactory constancy of the k's calculated without the assumption of a by-reaction teaches this too) that equation (3) cannot render the facts. Two explanations seem possible to me, viz.:

1. The reaction products of the HNO_3 and the anhydride themselves nitrate too; a priori this is, however, not very probable. Of a nitration with tetranitromethane a record is, indeed, found in the literature [1]), but there either the benzene derivate itself was of a basic nature, or a basic substance (pyridine) was added to it. It is also possible that intermediately nitro acetic acid is formed, or another intermediate product; whether this could nitrate here I dare not decide. At any rate it would be exceedingly accidental that the effect should be exactly as if no by-reaction had taken place.

2. The presence of the benzene prevents the by-reaction; this would e.g. be conceivable when an addition compound was formed. In this case the reaction would, however, proceed monomolecularly; I have, therefore, calculated constants in some experiments according to the equation $k = \frac{2.3025}{t} \log \frac{B}{B-x}$ (B = initial concentration benzene, hence also that of the addition compound on excess of HNO_3); sometimes the k's were then actually in somewhat better agreement, but in the experiments performed with the greatest care, as in those of Table 1 this was very certainly *not* the case. It should be taken into account however that there will be an equilibrium between addition compound and its components; the reaction scheme would then become: $A + B \rightleftarrows A\,.\,B \rightarrow C + D$.

Now whichever explanation may be the true one, I thought it in any case desirable to determine besides the nitric acid, also the nitrobenzene formed, because I have then a direct answer to the question whether the HNO_3 is only abstracted by the benzene or also at the same time by the solvent. This has succeeded in the following way: from the sample received in lye from the reaction mixture, the nitro benzene was shaken out by *petroleum ether*, the solvent was distilled off at a fractionation column and the $C_6H_5NO_2$ was dissolved in alcohol. In this it was determined *titanometrically* according to KNECHT and HIBBERT[1] in the modified arrangement, as indicated by KOLTHOFF[2]. By analyses of solutions of $C_6H_5NO_2$ in anhydrous acetic acid of known content I have beforehand ascertained the practicality of this method. I will confine myself here to stating this, as I have the results, obtained in a real nitration experiment, not yet completely at my disposal.

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[1]) E. SCHMIDT und U. FISCHER, Tetranitromethan als Nitrierungsmittel, B. **53**, 1529 [1920].

Ionisation of the nitric acid. It is, of course, conceivable that the HNO_3 in the anhydrous acetic acid is partially ionized. In this case the *k*'s would have to be calculated from a more complicated equation, as the effective concentration of the HNO_3 is smaller than the concentration determined analytically. However WALDEN[3] has shown that the ionisation of acids in solvents is independent of the law of NERNST-THOMSON; thus ε of strong sulphuric acid is almost equal to that of water and yet the HNO_3 in these solvent is not split up into H^+ and NO_3^-; for, if this were the case, the reaction velocity constants found by MARTINSEN could not have been in such good agreement.

Reaction between anhydrous acetic acid and nitric acid with strong sulphuric acid as a catalyst. When once to a solution of HNO_3 in $(CH_3CO)_2O$ of the usual strength (about 0.75 n) I added a little strong H_2SO_4 with the intention of diminishing the NO_2-content in this way, I suddenly observed an almost explosive reaction. The temperature of the vehemently boiling liquid rose to about 122°. On repetition of the experiment I found that after dilution with water an oil separates, which appeared to be tetranitromethane; the aqueous layer rendered alkaline contained NH_3; the nitric acid must, therefore, have acted very strongly oxydizing, which also appears from the expulsion of nitric vapours. I further found that previous addition of *urea* prevents the reaction, or at least greatly retards it. If no H_2SO_4 is added, solutions of the concentration used can remain at room temperature for a long time without anything being perceived of a violent reaction. The catalytic influence of the H_2SO_4 in this reaction[4] has, as far as I have been able to verify, not been described in the literature as yet.

I have also made experiments in which the reaction flask was placed in a large dish filled with cold water, which was vigorously stirred in order to lead off the reaction heat quickly. For I hoped that the HNO_3 — at least partly — would react according to the equation $HNO_3 + (CH_3CO)_2O \rightarrow CH_3CONO_3 + CH_3COOH$.

In this way the acetyl nitrate could then be prepared without it being necessary to make N_2O_5 first. A fractional vacuum distillation to which I subjected the mixture, has not produced the desired result so far. If possible I will repeat this experiment.

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[1] E. KNECHT and E. HIBBERT, New Reduction Methods in Volumetric Analysis.

[2] I. M. KOLTHOFF, Rec. 45, 169 [1926].

[3] P. WALDEN, Trans. FARADAY Soc. 6, 71 [1910].

[4] Addition of 8—10 dr. per 50 cc. solution is sufficient.

Summary of the Results.

At this place it will be sufficient for me to summarize my other experiments shortly. At 18° good constants were also obtained, and this over a wide conversion range; in one experiment k was found on an average 0.00128; from k_{18} and k_{25} the value 2.3 follows for the temperature coefficient between 18° and 25°; at 0° and at 40° the "constants" were bad; the k's decrease, resp. increase considerably here. — In an experiment in which the benzene was present in an excess of gramme molecules, no nitration took place; here the concentrations applied were, however, small; it is, therefore, desirable to repeat this experiment with e.g. $[C_6H_6] = 0.75$ and $[HNO_3] = 0.5$. — In another experiment the HNO_3 was about 0.5 molar, the C_6H_6 about 0.33. Here, too, constants were obtained but, very remarkably, k_{25° was here only 0.00166 instead of 0.00232, as in the experiment of Table 1. By measurement of the initial concentrations of the HNO_2 it will have to be decided whether they are possibly in (linear) relation to the reaction constants. Intentional addition of HNO_2 must render k greater then. — In glacial acetic acid *no* nitration takes place (or at least an exceedingly slow one): After 24 hours the titer of the nitric acid is still unchanged. Addition of anhydride to bind the water in the glacial acetic acid did not change this at all.

In the experiments which I still hope to make, I shall in particular devote my attention to the rôle of the concentration of the nitric acid in connection with the initial concentration of the HNO_2. It is further my intention to carry out a few more measurements with acetyl nitrate, both in anhydrous acetic acid and in *tetrachloro-carbon*. This is possible, as there is now no water-abstracting agent necessary. — In strong sulphuric acid the nitration proceeds very rapidly, in ether on the other hand, very slowly; it is, therefore, possible that these substances, applied in suitable mixing proportion, form a solvent in which the nitration of benzene, chloro-benzene etc. proceeds with measurable velocity. Also in this direction it will perhaps be tried to arrive at some results.

Amsterdam, 23. 6. '28. *Physiol. Lab. of the University.*

Mathematics. — *The Representations of a Linear Ray Complex on Point-Space that Associate Quadratic Surfaces to the Bilinear Congruences of the Complex.* By G. SCHAAKE. (Communicated by Prof. HENDRIK DE VRIES.)

(Communicated at the meeting of March 31, 1928).

§ 1. Through the well known representation of the lines l of space on the points P of a quadratic variety V_4 in R_5, to a linear complex C a quadratic variety V_3 lying in a four-dimensional space R_4 is associated. A representation of the lines l of C on the points L of space associates a point L of R_3 to any point P of V_3 and inversely.

Through the one-one correspondence (P, L) the spacial sections Φ of V_3, on which the bilinear congruences of C are represented, are transformed into surfaces Ω of point-space. These ∞^4 surfaces Ω form a linear system Σ. Indeed, four points of V_3 define a spacial section Φ so that four points L of space define one surface Ω.

To the surfaces Φ of V_3 lying in the spaces of a pencil with base-plane β, there correspond the surfaces Ω of a pencil in Σ. The base curve of this pencil consists of the curve associated to the intersection βV_3, i.e. of the image of a quadratic scroll of C, and of one or more possible base curves of Σ.

If to any space R_3 of R_4 we associate the surface Ω of Σ that corresponds to the intersection Φ of R_3 and V_3, we get a collinear correspondence between the spaces R_3 of R_4 and the surfaces Ω of Σ.

Consequently to the spaces R_3 through a line r of R_4 there correspond the surfaces of a net of Σ. Besides the fixed base curves and base points of Σ the surfaces of this net have two variable points in common, the images of the points of intersection of r and V_3.

To the spaces R_3 through a point P of V_3 there are associated the surfaces Ω of a linear complex in Σ. Besides the base curves and base points of Σ these surfaces have the point L in common that corresponds to P.

In order to find representations of a linear complex C on three-dimensional point-space, we must, therefore, in the first place determine linear systems of ∞^4 surfaces with such base curves and base points that three arbitrary surfaces of the system have two more points in common.

We shall now start from such a system Σ and we suppose a collinear correspondence to exist between the surfaces Ω of Σ and the spaces R_3 of R_4. A point L of space carries ∞^3 surfaces Ω, which form a linear complex. To these surfaces Ω there correspond the spaces R_3 of R_4 through a point P of R_4. If we associate this point P to L, the points P form a three dimensional quadratic variety V_3 in R_4. For the image points L of the points P of the intersection Φ of a space R_3 and V_3 form the surface Ω associated to R_3. Further the surfaces Ω corresponding to the spaces R_3 through a line r, have two points outside the base curves and base points of Σ in common, so that the surfaces Φ in the said spaces R_3 have two points in common. Accordingly a line r of R_4 cuts V_3 in two points.

Now if the lines l of a linear complex C can be represented on the points P of V_3, by representing each line l on the point L that is associated to the point P corresponding to l, we find a representation of the lines l of C on the points L of space.

We can find systems Σ by the aid of the remark that the surfaces of such a system that contain a point of Σ outside the base curves and base points, form a homaloid complex. Inversely from a homaloid complex which contains one or more isolated base points, we can deduce a system Σ by omitting one of these base points.

§ 2. We find three systems Σ of quadratic surfaces, which, perhaps, can lead to a representation of a linear complex on point-space. They consist of:

1. the quadratic surfaces through a conic k^2;
2. the quadratic surfaces through a line q and two points O_1 and O_2;
3. the quadratic surfaces through three given points O, O_1 and O_2 that touch a plane ω at O [1]).

§ 3. We shall first choose for Σ the system of the ∞^4 quadratic surfaces that contain a given conic k^2, and we shall suppose a collinear correspondence to exist between the surfaces Ω of Σ and the spaces R_3 of R_4. In this way we get a one-one correspondence between the points L of space and the points P of a quadratic variety V_3 in R_4.

The plane α of k^2 forms a surface Ω with any plane of space. The ∞^3 surfaces Ω that consist of α and an arbitrary plane of space, form a linear complex belonging to Σ, the surfaces of which have all points of α in common. To these surfaces correspond the spaces R_3 through a point A of V_3. This point is a cardinal point for our representation; the associated points L form the plane α.

To the spaces R_3 through a line a containing A there correspond the surfaces Ω consisting of α and the planes of a sheaf. The vertex of this sheaf is the point L corresponding to the point of intersection of a and V_3 different from A. Hence A is a single point of V_3.

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[1]) Cf. STURM, Geometrische Verwandtschaften, IV, § 134.

Any point L of k^2 is singular for our representation as any spacial section Φ must contain a point associated to L. Any space R_3 contains one point of the locus of the points that are associated to L; hence the points corresponding to L form a line k. This line passes through A, because L belongs to α.

The intersection of two surfaces Ω to which two spacial sections Φ are associated, i.e. a conic ω^2 that cuts k^2 twice, is the representation of a plane section φ^2 of V_3. As a conic ω^2 has two points in common with k^2, a plane of R_4 cuts the locus of the points P that correspond to the points L of k^2, twice. Accordingly the locus of the lines k of V_3 through A is a quadratic surface; it is the quadratic cone $\varkappa$ along which the space of contact to V_3 at A cuts this variety.

Let us consider a line r through a point P of $\varkappa$. The intersections of V_3 and the planes through P are represented on conics ω^2 through the point of k^2 that corresponds to P. All the surfaces Ω corresponding to the spaces R_3 through P touch, therefore, a fixed tangent plane of $\varkappa$ at L and form a complex. A point outside the tangent plane defines a net of the complex of which the surfaces are associated to the spaces R_3 through a line r that contains P. As the net has one isolated base point, the line r has one point in common with V_3 outside P. Any point P of $\varkappa$ is, accordingly, a single point of V_3.

A point P of V_3 outside $\varkappa$ is always a single point, as the point L corresponding to P together with a point L' for which LL'' does not cut k^2, defines a net of surfaces Ω with two isolated base points so that through P we can always draw a line that cuts V_3 outside P.

Accordingly we have here a general quadratic variety V_3, which may always be considered as the image of a general linear complex C.

We have already seen that a quadratic surface Φ and a conic φ^2 of V_3 are resp. represented on a quadratic surface Ω through k^2 and a conic ω^2 that cuts k^2 twice.

A plane γ of space which forms a surface Ω with α, is the image of a quadratic surface of V_3 containing A. Inversely from a surface Ω associated to such a surface the plane α splits off so that there remains a plane γ as image plane.

A line c of space being the intersection of two planes γ is the image of a conic φ_2 of V_3 trough A. Inversely such a conic being the intersection of two surfaces Φ through A is represented on a line c.

A line f of V_3 that has one point in common with a surface Φ containing A, is represented on a line of space. As f cuts one of the generatrices of $\varkappa$, the image line has one point in common with k^2. Inversely a line cutting k^2 cuts a surface Φ in one point that is not singular for the representation so that this line is the image of a line of V_3.

For the representation of the lines l of C on the points L of space we now find the following properties. C contains one cardinal line a, to which all points of the plane α are associated. There is a conic k^2 of singular image points. To a point of k^2 there correspond the lines l of a plane pencil of C containing a. The lines l corresponding to the points of k^2, form the special bilinear congruence of the lines of C that cut a.

A plane pencil of C is represented on a line that cuts k^2, a scroll of C on a conic that cuts k^2 twice. To a bilinear congruence of C there corresponds a quadratic surface Ω through k^2, which becomes a cone when the congruence is special.

A line of point-space is the image of a scroll of C containing a, a plane of the point-space the image of a bilinear congruence of C containing a.

In this way we have arrived at the representation of NÖTHER-KLEIN of the rays of a linear complex on the points of space.

§ 4. Let us now choose for Σ the system of the quadratic surfaces that contain a line q and two points O_1 and O_2. We get again a one-one correspondence between the points L of space and the points P of a quadratic variety V_3 in R_4 by the aid of a collinear correspondence between the surfaces Ω of Σ and the spaces R_3 of R_4. The surfaces of Σ that contain the line $O_1 O_2$, form a linear complex. The point T of R_4 associated to this complex is a singular point of the said correspondence on V_3 to which the whole line $O_1 O_2$ is associated. To the spaces R_3 through a line r containing T there correspond the surfaces Ω of a net of Σ. All these surfaces contain the lines q and $O_1 O_2$ and accordingly besides the base elements of Σ they have only points of $O_1 O_2$ in common, so that the line r in consideration cuts the variety V_3 in T only. The point T is, therefore, a double point of V_3.

A line o_1 through O_1 which cuts q and, therefore, lies in the plane $O_1 q \equiv \omega_1$, has two fixed points in common with all quadratic surfaces. Ω. Hence there are ∞^3 surfaces Ω that contain a line o_1. They form a linear complex belonging to Σ. The corresponding P of R_4 is a singular point on V_3 of the correspondence to which the line o_1 is associated. The points P corresponding to the lines o_1 form a line s_1 of V_3, as any surface Ω contains one line o_1 and consequently any spacial section of V_3 one of the said points P.

In the same way it appears that there is a second singular line s_2 on V_3. To a point of s_2 there corresponds a line o_2 in the plane $O_2 q \equiv \omega_2$, passing through O_2.

Consequently the double point T of V_3 and the points of two crossing lines s_1 and s_2 are singular for our representation. To a point of s_1 or of s_2 there correspond resp. a line of (O_1, ω_1) or of (O_2, ω_2) and inversely.

A plane section φ^2 of V_3 is represented on the cubic section ω^3 different from q of two surfaces Ω. The ∞^6 curves ω^3 are the twisted cubics through O_1 and O_2 that cut the line q twice.

The point O_1 is a cardinal point of our correspondence; the points P corresponding to this point form a plane σ_1 as any plane section φ that is represented on a curve ω^3 through O_1, contains one point P corresponding to O_1. This plane σ_1 passes through T because O_1 belongs to $O_1 O_2$ and through s_1 as the images of the lines o trough O_1 must belong to the plane corresponding to O_1.

In the same way it appears that the point O_2 is a cardinal point for our correspondence, to which the plane $Ts_2 \equiv \sigma_2$ is associated.

Also the points of the lines q are singular for the correspondence (L, P). As any surface Ω passes through the line q, any spacial section Φ of V_3 contains one of the points P that are associated to a definite point L of q, so that to such a point L a line t is associated. The lines t associated to the points of q, form a quadratic scroll τ^2. For a curve ω^3 cuts the line q twice, so that a plane section φ^2 of V_3 cuts two lines t.

Any line t has one point in common with the line s_1, viz. the image of the line o_1 that passes through the point L corresponding to t. We see, therefore, that the line s_1 and also the line s_2 belong to the scroll σ^2 connected with τ^2. The quadratic surface Φ_1 containing τ^2 is the intersection of V_3 and the space (s_1, s_2).

The surfaces Ω corresponding to the spaces R_3 through a point P of Φ_1 form the complex of the surfaces of Σ that touch a plane λ through q at the point L that corresponds to the line t through P. If for λ we choose a definite plane through q and if we make L describe the line q, P describes a line of the scroll σ^2 connected with τ^2. For any surface Ω touches the plane λ at one point L of q and accordingly any spacial section Φ cuts the locus of the points P once.

In point-space we have two cardinal points O_1 *and* O_2 *to which the planes* $Ts_1 \equiv \sigma_1$ *and* $Ts_2 \equiv \sigma_2$ *are associated. The points of the line* q *are singular. To a point* L *of* q *there corresponds a line* t*. The lines* t *form the intersection* Φ_1 *of* V_3 *and the space* $(s_1\, s_2)$.

Among the surfaces Ω there are ∞^2 surfaces that are degenerate in a plane λ through q and a plane μ through O_1O_2. To the surfaces Ω that contain a fixed plane μ there correspond the spaces R_3 of a pencil of which the axial plane τ is the locus of the points of V_3 that are associated to the points of μ so that τ belongs entirely to V_3. The plane τ passes through T and the line t of τ^2 that is associated to the point of intersection of μ and q.

In the same way it appears that to a plane λ there corresponds a plane σ of V_3. This plane σ also passes through T because λ generally cuts the line O_1O_2 in a point that is not singular for the representation, and passes through the line s that corresponds to the plane λ in the way indicated above. Among the planes σ are the planes σ_1 and σ_2 that correspond resp. to the planes q_1O_1 and q_2O_2.

The variety V_3 is formed by the lines p that project the image surface

Φ_1 out of T. Such a line p that is the intersection of a plane τ and a plane σ, is represented on the line of intersection b of a plane μ and a plane λ, i.e. a generatrix of the bilinear congruence B of the lines that cut q and O_1O_2. Inversely a line of B as the intersection of a plane λ and a plane μ is the image of the line of intersection of a plane σ and a plane τ, hence of a line p of V_3 through T. The points of intersection of b with O_1O_2 and q are resp. associated to the point T on p and the point of intersection of p and Φ_1.

On the hypercone V_3 a special linear complex C with axis a can always be represented. For the representation (l, L) of the lines l of C on the points L of space that arise through combination of the correspondence (l, P) between C and V_3 and the correspondence (P, L), we find the following properties if we suppose the sheaves of rays and the fields of rays of C to be represented resp. on the planes σ and on the planes τ of V_3.

In C among the lines that are singular for the representation we find in the first place the axis a that is represented on the points of O_1O_2. Further there are two plane pencils (F_1, φ_1) and (F_2, φ_2) of singular lines l of C that together with a define sheaves of rays of C and of which the vertices F_1 and F_2 are, accordingly, points of a; φ_1 and φ_2 are arbitrary planes passing resp. through F_1 and F_2.

A line l of (F_1, φ_1) or (F_2, φ_2) is represented resp. on a generatrix of the plane pencil (O_1, ω_1) or (O_2, ω_2). To the rays l of (F_1, φ_1) and (F_2, φ_2) there correspond resp. the points L of the planes ω_1 and ω_2.

In the space of the points L there lie two cardinal points, viz. O_1 and O_2. The lines l of C that are associated to O_1 and O_2 resp. form the sheaves of rays F_1 and F_2, i.e. the sheaves that are defined by a and the plane pencil (F_1, φ_1) and by a and the plane pencil (F_2, φ_2).

Also the points of $q \equiv \omega_1\omega_2$ are singular to which belong the rays of the bilinear congruence that has a and $\varphi_1\varphi_2 \equiv f$ as directrices. To a point of q are associated the rays of a plane pencil of this congruence that defines a field of rays together with a, i.e. the rays of a plane pencil with vertex on f in a plane through a.

The quadratic scrolls of C, that are those which have a as directrix, to which the conics of V_3 are associated, are represented on the twisted cubics ω^3 through O_1 and O_2 that cut q twice.

Among the ∞^6 curves ω^3 there are ∞^5 which are degenerate in a conic ω^2 through O_1 and O_2 and a line ω that cuts ω^2 and q in different points. These composite curves ω^3 correspond to the intersections of V_3 and the tangent planes of this variety so that to ω^2 as well as to ω a plane pencil in C is associated. As ω^2 lies in a plane through $O_1 O_2$, to which there corresponds a plane τ of V_3, this conic is the image of a pencil w, in a plane through a and the line ω that lies in a plane through q, represents a plane pencil w' with vertex on a. A conic ω^2 and a line ω that cut each other, are the images of two plane pencils

w and w' that have a line in common so that the plane of w passes through the vertex of w'.

A plane pencil of C in a plane through a is represented on a conic ω^2 *through* O_1 *and* O_2 *cutting q, a plane pencil of C with vertex on a on a line* ω *cutting q.*

To a plane pencil containing a there corresponds a line b of the bilinear congruence B that has q and $O_1 O_2$ as directrices. Such a line forms a conic ω^2 together with $O_1 O_2$.

If the vertex of w lies in φ_1 a generatrix of (O_1, ω_1) splits off from ω^2 and, accordingly, there remains a line through O_2. In the same way it appears that a plane pencil ω with vertex in φ_2 is represented on a line through O_1.

A sheaf of C with vertex on a is represented on a plane λ through q, a field of rays of C, of which, accordingly, the plane passes through a, on a plane μ through $O_1 O_2$. The image points L of the rays l of a bilinear congruence of C, of which, accordingly, one of the directrices coincides with a, form a quadratic surface Ω through O_1, O_2 and q.

If the directrix different from a lies in φ_1, the plane ω_1 splits off from Ω and, accordingly, there remains a plane through O_2. In the same way it appears that a bilinear congruence of C with directrix in φ_2 is represented on a plane through O_1.

A line g of points L is the image of a quadratic scroll γ of C. For g has two points in common with a surface Ω and, accordingly, γ contains two generatrices that cut an arbitrary straight line. As g cuts one generatrix of each of the plane pencils (O_1, ω_1) and (O_2, ω_2), γ has a generatrix in common with each of the plane pencils (F_1, φ_1) and (F_2, φ_2).

When g cuts the line $O_1 O_2$ g lies in a plane μ. In this case g is the image of the system of the tangents to a conic that touches a and the planes φ_1 and φ_2.

A plane α of point-space is the image of a congruence A (2,1) of C, because a conic ω^2 has two points in common with α and a line ω one point. The plane α cuts all generatrices of the plane pencils (O_1, ω_1) and (O_2, ω_2) and also the line $O_1 O_2$. Consequently A (2,1) contains the plane pencils (F_1, φ_1) and (F_2, φ_2) and also the line a. The congruence consists of the lines that cut a and touch a cone that has a as tangent and the planes φ_1 and φ_2 as tangent planes so that the vertex of the cone lies on the line $\varphi_1 \varphi_2$.

Let us now in point-space choose a curve k of the order n that has an o_1-fold point in O_1, an o_2-fold point in O_2 and that cuts the line q k times. Such a curve cuts a surface Ω, the planes ω_1 and ω_2 and a plane λ resp. in $2n - o_1 - o_2 - k$, $n - o_1 - k$, $n - o_2 - k$ and $n - k$ points that are not singular for the representation. Hence this curve is the image of a scroll of the degree $2n - o_1 - o_2 - k$ that contains resp. $n - o_1 - k$ and $n - o_2 - k$ generatrices of (F_1, φ_1) and (F_2, φ_2) and has a as $(n-k)$-fold directrix. If the curve has p more points in common with O_1O_2, the line a is besides a p-fold torsal line of the corresponding scroll.

We find the image curve of a scroll of C of the degree ν that has resp. ω_1 and ω_2 generatrices in common with (F_1, φ_1) and (F_2, φ_2) and of which a is a $\varkappa$-fold directrix by solving the quantities n, o_1, o_2 and k out of the equations

$$\begin{array}{ll} 2n - o_1 - o_2 - k = \nu & \\ n - o_1 - k = \omega_1 & \\ n - o_2 - k & = \omega_2 \\ n - k & = \varkappa. \end{array}$$

We find that the image curve is of the order $\nu + \varkappa - \omega_1 - \omega_2$, has resp. a $(\varkappa - \omega_1)$- and a $(\varkappa - \omega_2)$-fold point in O_1 and O_2 and cuts q $\nu - \omega_1 - \omega_2$ times.

For an arbitrary cone of C of the degree n we have $\nu = n$ and $\varkappa = \omega_1 = \omega_2 = 0$. Hence such a cone is represented on a curve of the order n that cuts q n times. It lies in a plane through q, as all generatrices of the cone pass through the same point of a.

To a curve of the n^{th}-class lying in a plane through a there corresponds a curve of the order $2n$ in a plane through O_1 O_2 with n-fold points in O_1, O_2 and on q.

We shall further consider a surface of the degree m that has a v_1-fold point in O_1 and a v_2-fold point in O_2 and of which q is an r-fold line. Such a surface has resp. $2m - v_1 - v_2 - r$, $m - r$, $m - v_1 - r$ and $m - v_2 - r$ points that are not singular for the representation in common with a conic ω^2, a line ω, a generatrix of (O_1, ω_1) and a generatrix of (O_2, ω_2). Hence the said surface is the image of a congruence $(2m - v_1 - v_2 - r, m - r)$ of which the rays of (F_1, φ_1) and (F_2, φ_2) are resp. $(m - v_1 - r)$- and $(m - v_2 - r)$-fold lines.

As the surface cuts $O_1 O_2$ in $m - v_1 - v_2$ points outside O_1 and O_2, a is an $(m - v_1 - v_2)$-fold line of the congruence.

The image surface of a congruence (μ, ϱ) that has the rays of (F_1, φ_1) and (F_2, φ_2) resp. as φ_1- and φ_2-fold lines, is of the degree $\mu + \varrho - \varphi_1 - \varphi_2$; it has a $(\varrho - \varphi_1)$-fold point in O_1, a $(\varrho - \varphi_2)$-fold point in O_2, and q is a $(\mu - \varphi_1 - \varphi_2)$-fold line of the image surface.

We get a representation of the kind that has been investigated, in the following way. We choose two points O_1 and O_2 on the axis a of C. Further we consider two planes φ_1 and φ_2 that are cut by $O_1 O_2$ resp. in the points F_1 and F_2.

A line l of C that cuts φ_1 and φ_2 resp. in S_1 and S_2, is represented in the point of intersection L of the lines $O_1 S_2$ and $O_2 S_1$.

This representation has been treated by me in the "Nieuw Archief voor Wiskunde", 2$^{\text{de}}$ reeks, deel XIV, p. 330.

If we suppose that to the sheaves and fields of rays of C there correspond resp. the planes τ and the planes σ, we get a representation that arises from the one we have investigated if in stead of C we choose the reciprocal figure.

§ 5. Finally we choose for Σ the ∞^4-system of the quadratic surfaces Ω that pass through three given points O, O_1 and O_2 and touch a given plane ω at O. We shall first investigate the one-one correspondence between the points P of space and the points L of a quadratic variety V_3 in R_4 which we get by the aid of a collinear correspondence between the surfaces Ω of Σ and the spaces R_3 of R_4.

The plane $OO_1 O_2$ contains a pencil of conics σ^2 that pass through O_1 and O_2 and touch the plane ω at O. Each of these conics has four fixed points in common with the surfaces Ω so that the surfaces Ω containing a conic σ^2 form a linear complex. To this a point S_2 of V_3 is associated, which is, accordingly, singular for our correspondence. The locus of the points S_2 is a line s_2 of V_3, as any surface Ω contains one conic σ^2 and, therefore, in any spacial section Φ of V_3 there lies one point S_2.

To a plane α of V_3 there corresponds a biquadratic surface ψ^4 of V_3. For to a plane section φ^2 of V_3 an intersection k^4 of two surfaces Ω is associated, which cuts α in four points so that a conic φ^2 has four points in common with the surface corresponding to α.

Because any conic σ^2 cuts a plane α twice, all surfaces ψ^4 have the line s_2 as double line.

A generatrix o of the plane pencil (O, ω) touches all surfaces Ω at O. The surfaces Ω through o form, therefore, a linear complex. Hence to a line o there corresponds a point S_1 of V_3 that is singular for the correspondence (l, P). As any surface Ω contains two lines o, any spacial section of V_3 has two points in common with the locus of the points S_1. Accordingly the points S_1 form a conic $s_1{}^2$. All surfaces ψ^4 pass through $s_1{}^2$.

The generatrix of (O, ω) in the plane $OO_1 O_2$ forms a conic σ^2 together with O_1O_2. To the linear complex of the surfaces Ω that contain this conic σ^2, there corresponds a point P which lies on s_2 as well as on $s_1{}^2$. Hence we see that s_2 and $s_1{}^2$ have one point S_1 in common.

The singular points of the correspondence (P, L) *on* V_3 *form a line* s_2 *and a conic* $s_1{}^2$ *that have a point* S *in common. To a point of* s_2 *there corresponds a conic* σ^2 *in the plane* $OO_1 O_2$, *to a point of* $s_1{}^2$ *there corresponds a generatrix of the plane pencil* (O, ω).

The linear complex of the surfaces Ω through the conic σ^2 consisting of the lines OO_1 and OO_2, is formed by the quadratic cones with vertex in O that have OO_1 and OO_2 as generatrices. To this complex a point T of s_2 is associated. The surfaces Ω corresponding to the spaces R_3 that contain a line through T, form a net of cones with vertices in O that have the generatrices OO_1 and OO_2 in common. As this net does not generally contain any base points outside OO_1 and OO_2, an arbitrary line through T has no point in common with V_3 besides T. Accordingly the variety V_3 is a hypercone with vertex T.

When the cones of the said net have a line different from OO_1 and OO_2 in common, the corresponding line through T lies entirely on V_3. Also the inverse holds good. Hence a line of V_3 through T is represented on a line through O and inversely.

The cones Ω are the images of the quadratic cones in which the three-dimensional spaces through T cut V_3. Among the ∞^3 cones Ω there are ∞^2 degenerations, each of which is formed by a plane through OO_1 and a plane through OO_2. They correspond to the intersections of V_3 and the tangent spaces of this hypercone. The planes through OO_1 are associated to the planes σ of one of the systems of planes, the planes through OO_2 correspond to the planes τ of the other system of V_3.

A line s of V_3 in a plane σ is the intersection of this plane and a spacial section Φ of V_3. Consequently to such a line s there corresponds the intersection of a plane through OO_1 and a surface Ω, i.e. a conic s^2 through O and O_1 that touches ω at O. In the same way it is evident that to a line t of V_3 in a plane τ a conic t^2 is associated that passes through O and O_2 and touches ω at O.

As any base curve k^4 of a pencil of the complex Σ passes through O_1, any plane section φ^2 of V_3 contains one point corresponding to O_1. Accordingly O_1 is a cardinal point for our representation and the points corresponding to O_1 form a plane of V_3. This plane passes through s_2 because all conics σ^2 contain the point O. It is the plane τ through s_2, for it has no point different from T in common with an arbitrary plane τ that is represented on a plane through OO_2. We shall call the plane of V_3 corresponding to O_1 τ_1.

In the same way it appears that O_2 is a cardinal point for the correspondence (P, L) and that to this point the plane σ_1, the plane σ through s_2, is associated.

Any curve k^4 has a double point in O, because the surfaces Ω touch each other at O. Hence two of the points corresponding to O lie in an arbitrary plane section φ_2 of V_3. The point O is, therefore, a cardinal point and the points corresponding to O form a quadratic surface. As the conics σ^2 and the lines o all pass through O, the quadratic surface associated to O must contain the conic $s_1{}^2$ and the line s_2. It is, therefore, the intersection of V_3 and the space through $s_1{}^2$ and s_2, that is a quadratic cone $\varkappa$ with vertex T.

Consequently the space of the points L contains three cardinal points, viz. O, O_1 and O_2. The points of V_3 corresponding to O, O_1 and O_2 form resp. the quadratic cone $\varkappa$ that projects $s_1{}^2$ out of T, the plane τ_1 passing through s_2 and the plane σ_1 containing s_2.

An arbitrary special linear complex C with axis a can always be represented on the hypercone V_3. We shall suppose that this representation associates the planes σ to the sheaves of C and the planes τ of V_3

to the fields of C. In this case we find the following properties of the representation (l, L) that arises through combination of the correspondences (l, P) and (P, L).

The linear complex C contains a plane pencil (A, α) of singular lines l one of which is a. The image points of an arbitrary generatrix of (A, α) form a conic σ^2 in the plane $O O_1 O_2$ that passes through O, O_1 and O_2 and touches ω at O. Accordingly to the lines of (A, α) there correspond the conics of a pencil Σ. The image points of the lines of (A, α) form the plane $O O_1 O_2$. The conic σ^2 corresponding to a is formed by the lines $O O_1$ and OO_2. Let the conic σ^2 that is formed by the line $O_1 O_2$ and the generatrix in $O O_1 O_2$ of the plane pencil (O, ω), be associated to the line c.

Further C contains a scroll σ^2 of singular rays that have the line c in common with (A, α). The image points of a generatrix of ϱ^2 form a ray of (O, ω). To the lines of ϱ^2 the points of the plane ω are associated.

In the space of the image points L we have three cardinal points, viz. O, O_1 and O_2. The lines l associated to O form the special bilinear congruence K with directrix a consisting of the generatrices of the plane pencils each of which is defined by a and a generatrix of ϱ^2. To O_1 there correspond the rays of the plane α, to O_2 the rays of the sheaf A.

A plane pencil w_1 of C with vertex on a is represented on a conic s^2 through O and O_1 that touches ω at O, a plane pencil w_2 of C in a plane through a on a conic t^2 through O and O_2 that touches ω at O. To a plane pencil w containing a a line through O is associated. Such a line forms a conic s^2 with $O O_1$, a conic t^2 with $O O_2$.

If we have a plane pencil w_1 of C containing a line of ϱ^2 of which the plane is a plane of contact that does not pass through a of the quadratic surface defined by ϱ^2, a generatrix of (O, ω) splits off from the image conic s^2; such a plane pencil is, therefore, represented on a line through O_1. In the same way it appears that a plane pencil w_2 of C of which the vertex lies on the quadratic surface defined by ϱ^2 but not on a, is represented on a line through O_2.

To a sheaf of rays of C, the vertex of which, accordingly, lies on a, there corresponds a plane through $O O_1$, to a field of rays of C, the plane of which, accordingly, passes through a, there corresponds a plane through OO_2. A bilinear congruence of C is represented on a quadratic surface Ω through O, O_1 and O_2 that touches ω at O. In particular the ∞^3 special bilinear congruences with vertex a are represented on the quadratic cones that contain $O O_1$ and $O O_2$. If the rays of (A, α) belong to such a special linear congruence the plane $O O_1 O_2$ splits off from the image cone and there remains a plane through O.

To a bilinear congruence of C that contains the scroll ϱ^2 and of which, accordingly, the directrix different from a belongs to the scroll connected with ϱ^2, the same as a, there corresponds a plane through $O_1 O_2$, as in this case the plane ω has split off from the image surface Ω.

A line g of points L has two points in common with a surface Ω, and cuts one conic σ^2 and one generatrix of (O, ω). A line g is, therefore, the image of a quadratic scroll γ^2 of C that has one generatrix in common with the plane pencil (A, a) and also with ϱ^2.

To a line g which cuts O_1O_2 there corresponds a scroll γ^2 through c that has one more generatrix in common with ϱ^2. The scrolls γ^2 that touch ϱ^2 along c, are represented on the lines through the point of intersection of O_1O_2 and ω.

If we have a line g that cuts OO_1, this lies in a plane through OO_1, so that all the generatrices of γ^2 belong to a sheaf C. In this case the common line of γ^2 and (A, α) coincides with a. A line cutting OO_1 is, therefore, the image of a cone containing a that touches α and contains the generatrix of ϱ^2 passing through its vertex. In the same way we see that a line g cutting OO_2 is the image of the system of tangents to a conic touching a at A that touches the generatrix of ϱ^2 lying in its plane.

A plane α of the points of space cuts a conic t^2, a conic s^2 and a conic σ^2 twice and a line of (O, ω) once. Such a plane is, therefore, the image of a congruence $\Gamma(2,2)$ that has the lines of (A, α) as double lines and contains the generatrices of ϱ^2.

If the plane passes through O_1, the field of rays α splits off from Γ and there remains, therefore, a congruence (2,1) containing the generatrices of (A, α) and the lines of ϱ^2. This congruence consists of the lines that cut a which touch an enveloping cone with vertex in α of the quadratic surface defined by ϱ^2, with the exception of the lines of α. In the same way it appears that a plane through O_2 is the image of the congruence (1,2) of the lines that cut a and a conic through A of the quadratic surface defined by ϱ^2 in different points.

We shall now consider a curve k^n of the order n that has resp. an o-, o_1- and o_2-fold point in O, O_1 and O_2. Let us suppose that r of the o branches through O of k^n touch ω at this point. The chosen curve cuts a surface Ω in $2n - o - o_1 - o_2 - r$ and a plane through OO_1 in $n - o - o_1$ points that are not singular for the representation and it cuts $n - o - o_1 - o_2$ conics σ^2 and $n - o - r$ lines of (O, ω) outside the base points of Σ.

Consequently the curve k^n is the image of a scroll λ belonging to C of the degree $2n - o - o_1 - o_2 - r$ that has a as $(n - o - o_1)$-fold directrix and has resp. $n - o - o_1 - o_2$ and $n - o - r$ generatrices in the plane pencil (A, α) and the scroll ϱ^2. As a plane through OO_2 cuts the curve in $n - o - o_2$ points that are not singular for the representation, the scroll λ has $n - o - o_2$ lines different from a in common with a field of rays containing a. Now such a field of rays contains in all $n - o_2 - r$ generatrices of λ. The line a is, therefore, an $(o-r)$-fold torsal generatrix of λ. The cuspidal points together with the planes of contact at the corresponding torsal lines define plane pencils that are represented on the tangents of k^n at O to the $o-r$ branches that do not touch ω. The r generatrices of (O, ω) that touch k^n, correspond to lines of ϱ^2, which, together with a, define plane pencils in each of which there lies a line of λ.

If k^n cuts the line OO_1 in one more point, a is a torsal generatrix of λ with α as corresponding torsal plane. For the cuspidal point corresponds to the plane through OO_1 that touches k^n at the point of intersection, and the torsal plane to the plane through OO_2 and the point of intersection, i.e. the plane OO_1O_2. If k^n has one more point in common with the line OO_2, we find that a is a torsal generatrix of λ with A as corresponding cuspidal point. If k^n touches the plane OO_1O_2 at O, a is a torsal generatrix of λ with A as corresponding cuspidal point and α as corresponding cuspidal plane.

Let us now investigate the image curve of a scroll of C of the degree ν that has a as α_1-fold directrix and as α_2-fold torsal generatrix, and that has resp. ω and ϱ generatrices in the plane pencil (A, α) and the scroll ϱ^2.

We find that such a scroll is represented on a curve of the order $2\nu-\varrho-2\omega-\alpha_2$, that has a $(\nu-\varrho-\omega)$-fold point in O, a $(\nu-\omega-\alpha_1-\alpha_2)$-fold point in O_1, an $(\alpha_1-\omega)$-fold point in O_2 and $\nu-\varrho-\omega-\alpha_2$ branches through O that touch ω at that point. This result only holds good when the α_2 cuspidal points and torsal planes corresponding to a are different resp. from A and α. If this is not the case the peculiarities of the image curve can be easily indicated by the aid of what has been found in the preceding paragraph.

If as a special case we choose a cone C of the n^{th}-degree, we have $\nu=n$ and $\alpha_1=\alpha_2=\omega=\varrho=0$. Such a cone is, accordingly, represented on a curve of the order $2n$ lying in a plane through OO_1 that has n-fold points in O and O_1. This curve touches itself at O as the n branches through this point all touch ω.

For a curve of the class n lying in a plane through a we have $\nu=\alpha_1=n$ and $\alpha_2=\varrho=\omega=0$. The system of tangents to such a curve is accordingly represented on a curve of the order $2n$ in a plane through OO_2 that has n-fold points in O and O_2 and of which the n branches through O_2 touch ω.

A surface of the degree m that has a p-fold point in O and q leaves touching ω at O, and for which O_1 and O_2 are resp. p_1-fold and p_2-fold points, has resp. $2m-p-p_2-q$, $2m-p-p_1-q$, $2m-p-p_1-p_2-q$, $m-p-q$ and $m-p$ points that are not singular for the representation, in common with a conic t^2, a conic s^2, a conic σ^2, a line of (O, ω) and a line passing through O. Such a surface is, therefore, the image of a congruence $(2m-p-p_2-q,\ 2m-p-p_1-q)$, for which the lines of (A, α) and of ϱ^2 are resp. $(2m-p-p_1-p_2-q)$- and $(m-p-q)$-fold lines and of which a plane pencil through a contains $m-p$ lines different from a.

If inversely we have a congruence (μ_1, μ_2) for which the lines of (A, a) are $\varkappa$-fold lines and the lines of ϱ^2 ϱ-fold lines, and that has π lines different from a in common with a plane pencil containing a, we find that this congruence is represented on a surface of the degree $\mu_1 + \mu_2 - \varkappa - \varrho$ that has a $(\mu_1 + \mu_2 - \varkappa - \varrho - \pi)$-fold point in O, $\pi - \varrho$ leaves touching ω at O and resp. a $(\mu_1 - \varkappa)$- and a $(\mu_2 - \varkappa)$-fold point in O_1 and O_2.

Mathematics. — *On a Configuration of Ten Lines in Space.* By G. SCHAAKE. (Communicated by Prof. R. WEITZENBÖCK.)

(Communicated at the meeting of June 30, 1928).

In his paper: "Ueber eine Configuration von zehn Geraden im projectiven R_3" [1]) Prof. R. WEITZENBÖCK has shown that there are configurations of five lines $l_1, \ldots, l_5$ in space so that any four lines of such a configuration have only one transversal and the five lines $l_1, \ldots, l_5$ do not all cut the same straight line. In this paper we shall give another derivation of this result.

As the five lines $l_1, \ldots, l_5$ do not cut the same line, they all belong to a general linear complex. We imagine, therefore, such a complex C and we make use of the representation of NOETHER-KLEIN of the lines l of this linear complex on the points P of a three-dimensional space R_3. We arrive in the easiest way at this representation if we start from the well known representation of the lines l of C on the points L of a general quadratic variety V_3 in R_4, choose a point L_1 of this variety and associate to any point Q of V_3 its projection P out of L_1 on an R_3 in R_4.

We shall now choose the line to which the point L_1 of V_3 is associated, as l_1. In this case l_1 has an infinite number of image points P, which form a plane α in R_3. α contains a conic k^2 that consists of singular image points P. For any point P of k^2 is the image of a plane pencil of C that contains l_1. An arbitrary plane pencil w of C is represented on a line g that cuts k^2 and also the inverse holds good.

The only transversal of the lines $l_2, \ldots, l_5$ in C is associated to itself relative to C and belongs, therefore, to C. Consequently $l_2, \ldots, l_5$ lie in a parabolic bilinear congruence of C that has a line l'_1 of C as directrix. Accordingly the image points $P_2, \ldots, P_5$ of $l_2, \ldots, l_5$ must belong to the cone $\varkappa$ that projects k^2 out of the image point P'_1 of l'_1.

A bilinear congruence of C containing l_1 is represented on a plane of R_3, a parabolic bilinear congruence of C containing l_1 on a plane that touches k^2. As any of the quadruplets (l_1, l_3, l_4, l_5), (l_1, l_2, l_4, l_5), (l_1, l_2, l_3, l_5) and (l_1, l_2, l_3, l_4) must belong to a parabolic bilinear congruence of C, the planes $P_3P_4P_5$, $P_2P_4P_5$, $P_2P_3P_5$ and $P_2P_3P_4$ all must touch k^2.

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[1]) These Proceedings Vol. 31, p. 133.

We shall call the point where the plane $P_3 P_4 P_5$ touches the conic k^2 and the point where the generatrix through P_2 of $\varkappa$ cuts the conic k^2 resp. R and S.

If we project out of P_2 on α, the intersection of $P_3 P_4 P_5$ and $\varkappa$ transforms into a conic c^2 that touches k^2 in R and S, and the projections C_3, C_4 and C_5 of P_3, P_4 and P_5 are the angular points of a triangle that is circumscribed about k^2 and inscribed in c^2.

If inversely we have a triangle $C_3 C_4 C_5$, that is circumscribed about k^2 and inscribed in a conic c^2 that touches k^2 twice, P_2 and the points of intersection P_3, P_4 and P_5 different from P_2 of $P_2 C_3$, $P_2 C_4$ and $P_2 C_5$ with $\varkappa$ form a quadruplet (P_2, P_3, P_4, P_5) of points that satisfies the above mentioned conditions and P_2, P_3, P_4 and P_5 are the image points of lines l_2, l_3, l_4 and l_5 of C which together with l_1 form a quintet $(l_1, l_2, l_3, l_4, l_5)$ of which we shall prove the existence.

Let us suppose a projective transformation in α that associates the isotropous points of this plane to R and S and two real points of α to the pole K of RS relative to k^2 and c^2 and a real point of k^2. Through this transformation k^2 passes into a real circle k' and c^2 into a circle c' that is concentric with k' and that must have an inscribed triangle which is circumscribed about k'.

If we call the points that are associated to C_3, C_4 and C_5 resp. C'_3, C'_4 and C'_5, the triangle $C'_3 C'_4 C'_5$ must be equilateral and the circle c' has a radius that is twice as large as that of k'.

Inversely from the circle c' defined in this way we derive one real conic c^2 touching k^2 in R and S by choosing on any line l through K the two points that are harmonically separated from K by one of the points of intersection of k and k^2 and the point of intersection of k and RS. In the conic c^2 constructed in this way an infinite number of triangles can be inscribed of which the sides touch k^2. Thus the announced proof has been given.

The three angular points of an equilateral triangle and an isotropous point considered as points of the circle circumscribed about the triangle, form an equianharmonic quadruplet. Consequently the same holds good for the points of the quartet (S, C_3, C_4, C_5) considered as points of c^2.

If, therefore, for C_3 we choose a real point of c^2, C_4 and C_5 will be conjugate imaginary points of this conic. Consequently at most three of the lines $l_1, \ldots, l_5$ are real. In this case also only three of the five transversals $l'_1, \ldots, l'_5$ are real. For these five transversals are represented on the vertex P' of $\varkappa$, on the point S and on the points of contact of the planes $P_2 P_4 P_5$, $P_2 P_3 P_5$ and $P_2 P_3 P_4$ with k^2.

Also at most three of the lines $l'_1, \ldots, l'_5$ can be real. This is a consequence of $l'_1, \ldots, l'_5$ likewise forming a quintet of the kind in question. For $l'_1, \ldots, l'_4$ have the line l_5 in C as transversal and not an infinite number as in this case the lines $l'_1, \ldots, l'_4$, hence also $l_1, \ldots, l_5$, would belong to the same scroll.

The lines P_2S, P_2P_3, P_2P_4 and P_2P_5 form an equianharmonic quartet if we consider them as generatrices of the cone that projects c^2 out of P_2. Consequently the same holds good for the lines P'_1P_2, P'_1P_3, P'_1P_4 and P'_1P_5 considered as lines of $\varkappa$, hence for the vertices and also for the planes of the plane pencils that are defined by l'_1 and the lines l_2, l_3, l_4 and l_5.

Accordingly the transversal of an arbitrary quartet of $(l_1, l_2, l_3, l_4, l_5)$ cuts the lines of this quartet in points that form an equianharmonic quartet.

If also for l_1 we choose a line of C that is not singular for the representation, our problem is equivalent to the definition of two quintets of points $(P_1, \ldots, P_5)$ and $(P'_1, \ldots, P'_5)$ such that the twenty lines $P_1P'_2, \ldots, P_5P'_4$ cut the conic k^2.

Finally we remark that through application of the theorem of PONCELET to the conics k^2 and c^2 we arrive at the following property: a scroll of a quadratic surface touching l_1 and l_2 does not generally contain any triplet of lines that forms a quintet $(l_1, \ldots, l_5)$ together with l_1 and l_2. If, however, such a scroll contains one such a triplet, it contains an infinite number of them.

Mathematics. — *On the Boundary Limit Infinite of a Holomorphic Function.* By Prof. J. Wolff. (Communicated by Prof. L. E. J. Brouwer).

(Communicated at the meeting of May 26, 1928).

In the Annales scientifiques de l'école normale supérieure, Vol. **42**, 1925, on pp. 147 sqq. N. Lusin and J. Priwaloff construct a function that is holomorphic inside the unity circle, with the property that its absolute value approaches to infinite in concentric rings with the unity circle as limit. On pp. 183 sqq. of the same paper they construct a holomorphic function $f(z)$ that approaches to infinite for $|z| \to 1$ along rays whose arguments form a set with measure 2π.

In this paper we shall show that in a simple way functions can be constructed that have both the above mentioned properties *at the same time*.

Suppose $$0 < r_1 < R_1 < r_2 < R_2 < \ldots < r_k < R_k \to 1 \quad . \quad . \quad . \quad . \quad (1)$$

Let further for any k: n_k be the least positive integer for which

$$\left(\frac{r_k}{R_k}\right)^{n_k} < \frac{1}{3k^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$A_k = \frac{3}{R_k^{n_k}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Then

$$\left.\begin{array}{lll} |A_k z^{n_k}| \geqq 3 & \text{for} & |z| \geqq R_k , \\ |A_k z^{n_k}| \leqq \dfrac{1}{k^2} & ,, & |z| \leqq r_k . \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It follows from this that the infinite product

$$f(z) = \prod_{k=1}^{\infty} (1 - A_k z^{n_k}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

converges uniformly for $|z| \leqq \varrho < 1$ for any ϱ that is less than 1, because beginning at a certain k the inequality $\varrho \leqq r_k$ holds good, hence also $|A_k z^{n_k}| \leqq \frac{1}{k^2}$. Accordingly $f(z)$ is holomorphic for $|z| < 1$.

Let us now consider the ring

$$G_n : \qquad R_n \leqq |z| \leqq r_{n+1} .$$

Here

$$f(z) = \prod_{k=1}^{\infty}(1 - A_k z^{n_k}) \,.\, \prod_{k=n+1}^{\infty}(1 - A_k z^{n_k}),$$

hence, according to (4)

$$|f(z)| > 2^n \prod_{k=n+1}^{\infty}\left(1 - \frac{1}{k^2}\right) \geqq 2^n \prod_{k=2}^{\infty}\left(1 - \frac{1}{k^2}\right) \to \infty \quad \text{for} \quad n \to \infty.$$

Consequently $f(z)$ approaches uniformly to ∞ in the rings G_n and that independently of the special choice of the range of numbers (1).

By choosing a number of the form $p!$ for n_k in stead of the least positive integer that satisfies (2), e.g. the least number $p!$, we obtain also here as in the example of LUSIN and PRIWALOFF, that $|f(z)| \to \infty$ for $|z| \to 1$ on a set of rays that is everywhere dense.

But by a special choice of the range (1) we can obtain that $f(z)$ has also the above mentioned second property.

If α is a root of $z^n - 1 = 0$ it is easily seen that

$$|z^n - 1| > \frac{p}{n} \quad \text{for} \quad |z - \alpha| = \frac{1}{n^2}, \quad n > 1,$$

where the positive number p does not depend on n.

Consequently if β is a root of $A_k z^{n_k} - 1 = 0$ we have

$$|A_k z^{n_k} - 1| > \frac{p}{n_k} \quad \text{for} \quad |z - \beta| = \frac{1}{n_k^2 A_k^{\frac{1}{n_k}}} = \frac{R_k}{n_k^2 3^{\frac{1}{n_k}}} = \varrho_k \,.$$

We have

$$\varrho_k \sim \frac{1}{n_k^2} \quad \text{for} \quad k \to \infty \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

We choose the range of numbers (1) thus: let θ be a constant between 0 and 1 of which the value is left undetermined and suppose

$$r_k = 1 - \theta^{2k} \quad , \quad R_k = 1 - \theta^{2k+1} \quad , \quad k = 1, 2 \quad . \quad . \quad . \qquad (7)$$

The roots of the binomium $A_k z^{n_k} - 1$ lie in the ring $r_k < |z| < R_k$ according to (4). Hence the distance of a root of $A_k z^{n_k} - 1 = 0$ to the circle $|z| = r_{k+1}$ is greater than $\theta^{2k+1} - \theta^{2k+2} = (1 - \theta)\, \theta^{2k+1}$.

In the same way its distance to the circle $|z| = R_{k-1}$ is greater than $(1 - \theta)\, \theta^{2k-1}$.

From (2) and (6) it follows that:

$$\varrho_k \sim \left\{ \frac{lg\,(1 - \theta)^{2k+1} - lg\,(1 - \theta)^{2k}}{lg\,(3\,k^2)} \right\}^2 \sim \frac{(1 - \theta)^2\, \theta^{4k}}{(lg\,k)^2} = o\,(\theta^{2k+1}) = o\,(\theta^{2k-1}).$$

Further for $k \to \infty$: ϱ_k is infinitely small relative to the distances of the roots of $A_k z^{n_k} - 1 = 0$ inter se, according to (6).

From all this it follows that the circles γ_k which have the roots of the polynomials $A_k z^{n_k} - 1$ as centres and ϱ_k as rays, for a sufficiently great k, all lie in pairs outside each other and inside the unity circle.

It is also evident that for a sufficiently great k the circle γ_k lies between the circles $|z| = r_{k+1}$ and $|z| = R_{k-1}$.

Now on circle γ_k we have

$$|f(z)| > 2^{k-1} \cdot \frac{p}{n_k} \cdot \prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2}\right).$$

And

$$\frac{1}{n_k} \sim \frac{(1-\theta)\,\theta^{2k}}{lg\,k}.$$

We now choose θ such that $2\,\theta^2 > 1$.

In this case the minimum of $|f(z)|$ on the n_k circles γ_k approaches to infinite for $k \to \infty$. As on the circles $|z| = R_{k-1}$ and $|z| = r_{k+1}$ the minimum of $|f(z)|$ approaches to ∞ for $k \to \infty$, the circles γ_k lie in the ring $R_{k-1} < |z| < r_{k+1}$ for any sufficiently great k, and all null points of $f(z)$ lie inside the circles γ_k, we have:

if from the interior of the circle $|z| < 1$ all circles γ_k are omitted beginning at a sufficiently great k, there remains a domain G where $|f(z)|$ uniformly approaches to ∞ for $|z| \to 1$.

The series $\Sigma \frac{1}{n_k}$ converges according to $\frac{1}{n_k} = O(\theta^{2k})$.

The sum of the angles at which out of O we see all the circles γ_k corresponding to a given k, is $O\left(\frac{1}{n_k}\right)$ according to (6).

Hence the angles at which we see all the circles γ_k out of O, form a convergent series. Let $\varepsilon > 0$ and N be chosen such that the sum of these angles corresponding to $k > N$ is less than ε. The rays out of O that do not cut any circle with index $> N$, have arguments that form a set with measure $> 2\pi - \varepsilon$. On these rays z lies in G as soon as $|z|$ lies sufficiently close to 1, hence $|f(z)| \to \infty$. Accordingly the rays on which $f(z) \to \infty$ for $|z| \to 1$, have arguments that form a set with measure $> 2\pi - \varepsilon$. As ε was arbitrary they form a set E with measure 2π.

E is the collection of an enumerable number of sets that are nowhere dense, hence of the first category of BAIRE.

Let us also notice that in any sector

$$S\,(\alpha < arg\, z < \beta)$$

$f(z)$ assumes any value in an infinite number of points. For if in S $f(z) \neq a$, it is easily seen that the function $\frac{1}{f(z)-a}$ which is holomorphic in S, would approach to zero in any point of the arch of $|z| = 1$ that is the boundary of S, according to the properties of $f(z)$ in the concentric rings of the circles $R_k \leqq |z| \leqq r_{k+1}$ and on the rays passing through the points of E. Hence we should have $f(z) \equiv 0$.

Utrecht, March 22, 1928.

Physiology.— *Diet and reproduction.* III. By G. GRIJNS, K. DE HAAN and J. A. VAN DER LOEFF.

(Communicated at the meeting of June 30, 1928).

Since our former communication [1]) a lot of publications have appeared on this subject and all authors come to the postulation of a special nutritive factor controling reproduction.

EVANS [2]) describes the sterility in male rats on various diets, each of which is able to make young rats grow normally and distinguishes four grades of sterility, based on the microscopical investigation of the testes. Wheat germ oil prevents sterility when added to the deficient diet.

EVANS and BURR [3]) purified the unsaponifiable matter of wheat germ oil to a yellow oily substance, of which a single dose of 5 mg. administered to females in the beginning of pregnancy suffices to enable them to produce vigourous young and to rear them even on an E-poor diet.

BARNETT SURE [4]) confirms the observations of EVANS and BISHOP [5]) about the resorption of fetus in sterile females, giving photograms of the uterus, and also the observation of DANIELS and HUTTON, that rats fed a diet of skimmed milk, dextrin, with yeastvitamin, codliver oil and ethereal wheat embryo extract where sterile, but that the addition of the mixture of potassium aluminium sulfate, sodium fluoride, sodium silicate and manganese sulfate to this diet was able to preserve fertility in four generations.

SURE [6]) also proves, that wheatgerm oil not only prevents sterility, but that it also possesses lactation promoting properties. The lactation factor is destructed by heating till 110° C., the sterility preventing not. The ethereal extract contains the active substances, which are not destroyed bij saponifying the fat in alcoholic KOH [7]).

The butterfat contains much less vitamin E. than wheatembryo, but 10 % butterfat in de dietary will suffise to keep up reproduction [8]). Milksecretion requires considerably more B-vitamin than optimum growth 1) and codliveroil is but poor in vitamin E 2).

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[1]) These Proceedings, Vol. **28**, p. 942 and **29**, p. 873.

[2]) H. M. EVANS: Invariable occurrence of male sterility with dietaries lacking fatsoluble vitamin E. Proc. Nat. Acad. o. Sciences U. S. A. **2**, p. 373, 1925.

[3]) H. M. EVANS and C. O. BURR: The antisterility vitamin: fatsoluble E. Proc. Nat. Acad. of Sciences U. S. A. **2**, p. 334.

[4]) BARNET SURE: Dietary requirements for reproduction VI. J. o. Biol. Chem. **69**, p. 41, 1926.

[5]) H. M. EVANS and K. S. BISHOP: J. o. Metabol. Research 1923, p. 213, 242.
id. id. J. Amer. Med. Assoc. 1923, **81**, p. 889.

[6]) B. SURE, Dietary requirements etc. VII. J. o. Biol. Chem. **69**, p. 53.

[7]) id. „ „ „ VIII. „ „ „ „ **74**. p. 37, 1927.

[8]) id. „ „ „ XI. „ „ „ „ **74**, p. 71, 1927.

Hogan and Harshaw 3), who could not confirm the findings of Evans and his cooperators in his former experiments, stated that after they substituted lard for the "crisco" (a dehydrogenated cottonseed oil product) his rats really grew sterile.

On a ration of casein 20, cornstarch 50, lard 10, codliveroil 5, dried yeast 9, cellulose 2, salts 4, males become sterile and females can not rear their young. After 8 months they too where sterile. B-vitamine gave no improvement. The rats did not suffer from anaemia.

Clayton stated, that unsaturated animal fats in the ration seriously interfere with reproduction 4).

Sure and Shilling 5) described beriberi in suckling rats, whose mothers were fed a B-vitamin lacking diet. They conclude that for lactation twice as much B-vitamin is necessary as for growth. The clinical symptoms they described, differ much from what I saw myself of beriberi in man and in avian polyneuritis.

Soon after we had initiated our experiments with yellow fat, we also began such with *Ostelin*, a codliver oil product manufactured by the english company Glaxe.

Young rats got the following mixture direct after weaning.

Casein purified	80	Hydrogenated fat	34
Saltmixture	25	Ostelin 2 % in oliveoil	20
Amylum oryzae	260	Marmite	25
Albumen e sangue	10	Decitrated lemonjuice	25

In the first part of the experiment they grew normally, but than slackened a bit. The males did not surpass 190 gm the females reached 150.

As may be seen from table 1 out of 9 females only one had a litter ; she died in parturition. The males on the contrary showed reproduction when mated with normally fed females.

It is impossible to explain this experiment and those with maïze-foods by a single fertility controling factor.

For if we would try to explain the maizefood experiments, by assuming that the males are more sensible to the lack or the scantity of a fertility controling factor in the maize than the females are, in the case of the ostelin diet, where the females are sterile, the males not, we should come to the

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1) Barnet Sure Dietary requirements etc. X. J. o. Biol. Chem. **74**, p. 55, 1927.

2) id. „ „ „ IX. „ „ „ „ **74**, p. 45, 1927.

3) A. G. Hogan and H. M. Harshaw: Some relations between fertility and the Composition of Diet. Research Bulletin Univ. Missoury N⁰. 94, 1926.

4) May M. Clayton: The influence of diets containing unsaturated animal fats on reproduction and lactation in the rat. Journ. o. Biol. Chem. **74**, Proc. Am. Soc. of Biologist 21th meeting, p. LXXIV, 1928.

5) B. Sure and S. J. Schilling: The production of beriberi in the nursing young of the albinorat on diets entirely satisfactory for growth. J. o. Biol. Chem. **74**, LXXIV, 1927.

TABLE 1. Ration XVI (Osteline).

Male	Female	Young	Weaned	Male	Female	Young	Weaned
(500)	519	0		(609)	646	6 1)	0
(517)	521	0		640	641	0	
518	(542)	13	8	643	634	0	
520	(542)	9	0	„	639	2	0
526	521	0		„	639	0	
527	519	0		„	634	0	
„	533	0		644	645	0	
„	(537)	9	7	„	638	0	
528	(543)	9	0	(770)	638	0	
(584)	519	0		„	641	0	
„	521	0		„	645	0	
(609)	641	0		„	(634)	4 2)	0

opposite view, for there we should suppose the females to be the more sensible.

In our second communication we already showed, that male fertility and the milksecretion are controled by different factors. That milksecretion and female fertility also depends on different factors is not yet proven by our experiments, but we think it probable. One might think, that from the same factor milksecretion required more than gestation. But if we consider the velocity of growth we are inclined to hold the opposite view.

The curves of growth as they are ordinarily given are simple weight curves. They show the total of weight of the animal on a given time.

But it seems more likely, that the required concentration of a food factor in a diet depends on the *intensity* of growth. Therefore we must consider the increase of weight with regard to the weight reached in every moment. If we suppose the growth-constant as invariable during a short time between two weighings, we come to a formula $K = (ln \,.\, G_1 — ln \,.\, G_2) : (t_1 — t_2)$ 3). Now K is not invariable, but we come to an approximate diagram, if we take the intervals of observation short. We therefore plotted out our weights on several ages, and calculated the average value of K for every two weeks.

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1) Mother died in parturition; found in utero.

2) Died in pregnancy.

Figures in parenthesis mean rats on normal rations.

3) Cf. SAMUEL BRODY: Growth and development. Univ. Missouri Agric. Exper. Station. Research Buletin. 97, 1927.

In chart 1 the growthcurve of males and females (A) and the intensity curve (B) are given. We see, that the growth-intensity (percentage rate)

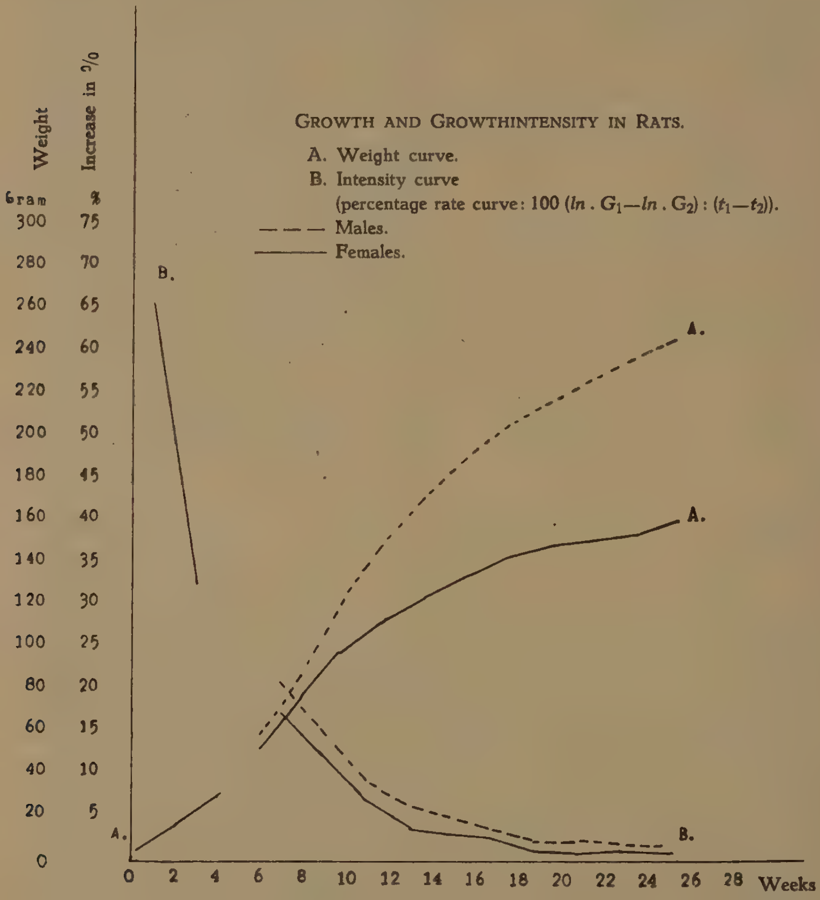

Fig. 1.

is diminishing very rapidly after birth. The curve becomes less and less steep with age.

The gap in the diagram between 4 and 6 weeks is caused by the fact, — that for the litters we had only weights of rats on experimental diets while the rats older than 5 weeks were all stockrats. So the figures of both series were not fully comparable, but as the curve shows the error is not great.

If we remember, that a rats gestation time is three weeks, and that a young rat weights about 4½ gr. while the ovum weighs only 25×10^{-4} m.gr.

and the secundinae have to be formed too, we see, that in utero the intensity of growth is still much greater.

From a number of females we microscopised the ovaria. They always contained Graafian folliculae and ovula. In two cases we found dead foetus in the whomb.

We have tried to get further information on the question of the number of fertility factors by investigating several diets. Firstly we sought for the constituent of normal food, in which the reproduction controling factors might be hidden. To that purpose we substituted in an experiment the suet by hardened fat. The results in table 2 show that it is not the suet that carries the fertilising factors.

TABLE 2. Stockration with hardened fat for suet.

First generation				Second generation			
Males	Females	Young	Weaned	Males	Females	Young	Weaned
813	851	5	4	936	964	9	8
	852	8	8		968	1	0
847	808	7	6	940	957	12	11
	807	11	0		958	5	0
Third generation				963	937	8	0
					999	10	10
1126	1130	8	7	1001	998	6	2
					1000	8	0

In a second experiment we gave water in the place of the milk, and withdrew clover and carrots. The young grew well; only the males did not grow as heavy as normal ones. One male mated with 2 females on stockdiet begot 17 young, 13 of which were weaned (77 %). Out of 7 females in 9 matings, 3 bore young, together 25 but none were weaned (table 3). The dropping of the milk from our normal ration diminished female fertility to the half, and made rearing impossible. Males showed fertile.

Maizefood I—III kept females fertile and permitted weaning; thus we should suppose the milkfactor either in coconutmeal and peanutmeal or in maize.

In the same time we continued our experiments on maize. We tried to find out firstly if the infertility of the males in the second generation was due to a faulty formation of the testes, or to a degeneration from long abstinence of indispensable foodstuff.

From 6 females and 3 males on maizefood I with codliveroil 2 % we had

TABLE 3. Stockdiet without milk.

Female	Male	Number of young	Weaned
806	836	0	—
„	800	0	—
809	836	10	0
„	800	4	0
816	836	4	0
823	836	0	—
825	833	0	—
„ [1]	914	5	0
826	914	0	—
(814)	800	9	5
(828)	800	8	8
(908)	914	7	0

20 male young and 15 female. (Communication I. p. 879) 4 males and 4 females kept the same diet as their eldren. The four males proved unfertile, their testes remained far below normal weight and in the epididymis no spermatoids were found (table 4 Nos 716, 750, 757, 762). This confirms our previous result (I p. 882). From the 4 females 2 were mated with males on stockrations and reared young. 10 males and the other 11 females got stockrations after weaning. 3 of the males were not regarded further. 3 (Nos 693, 701 and 702) were mated with several females and proved fertile (table 4). In the epididymis we found motile spermatozoids. The same was found in 3 others Nos 756, 749 and 760.

The remaining 6 males Nos 706, 708, 712, 713, 714, and 715 were fed after weaning maizefood I with codliveroil and 2 % from an acetone extract of unbolted wheatmeal. They were mated with normal fed females and all showed infertile. They had atrophy of testes and no spermatozoids in the epididymis. From the females used 4 procreated with normal males (table 5).

In a second series the elder rats already received maizefood I with cod liveroil and acetonextract. Here all males also proved sterile with atrophical testes (table 6, Nos 879, 883, 884, 895). In this case the females were less fertile than normal fed ones.

In the aceton extract of unbolted wheatmeal a fertility promoting factor could not be discovered.

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[1] From 3—5—'27 on stockration. 9—5 killed with gas. 5 foetus found in utero.

TABLE 4. Maizefood I with 2 % codliveroil.

Second generation						Remarks
Male	Female	Young	Weaned	Weight of testes	Spermatozoids	
716	705	0	—	0.73	—	
750	709	0	—	0.68	—	
757	710	0	—	0.92	—	Single tails ?
762	711	0	—	1.20	—	
693	(692)	6	0	3.32	++	This and next two males got stockration after weaning.
	(692)	2	0			
	(631)	1	0			
	(705)	0	—			
701	616	9	9	1.48	++	
	792	0	—			
702	704	8	6	2.56	++	
	608	8	0			
	658	5	0			
	799	5	0			
(783)	710	5	5		++	
	709	?				Abortus ?

TABLE 5. Maizefood I. with 2 % codliveroil and acetone extract of whole wheat meal.

Male	Female	Young	Weight of testes	Spermatozoids
706	(717)	0	0.27	—
708	(700)	0	1.12	—
712	(965)	0	0.68	—
714	(696)	0	0.75	—
714	(698)	0	0.79	—
715	722	0	0.84	—
(699)	772	11		
(785)	(700)	0		
„	(717)	8		
„	722	9		

TABLE 6. Same ration as table 5.

Male	Female	Young	Weaned	Weight of testes	Spermatozoids
879	902	0	—	not investigated	
883	(905)	0	—	1.04	—
884	882	0	—	0.87	—
„	885	0	—		
895	878	0	—	1.20	—
„	880	0	—		
(904)	882	0	—	3.25	++
„	885	0	—		
(922)	902	7	7		
(920)	878	3	0	3.09	++
„	880	0	—		

TABLE 7. Maizefood I with 2 0/0 codliveroil and 5 0/0 marmite.

First generation			Weaned	Weight of testes	Spermatozoids
Male	Female	Young			
697	737	4	4	well developed	
„	738	8	0		
„	745	0	—		
„	738	5	0		
„	738	4	0		
„	738	5	0		
„	(723)	9	9		
„	(726)	9	8		
779	741	7	7		
„	745	5	0		
910	827	0	—	3.15	++
Second generation					
868	906	0	—	1.21	—
„	906	0	—		
869	872	0	—	1.36	—
870	907	0	—	1.71	+
871	911 1)	0	—	1.21	+
876	913 1)	0	—	1.14	—

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1) 911 and 913 had earlier youngs.

TABLE 8. Maizefood I with 3 % butterfat substituted for hardened fat.

Male	Female	Young	Weaned	Weight of testes	Spermatozoids
First generation					
606	618	9	6		
"	622	10	10		
607	617	5	5		
"	621	9	0		
609	620	8	7		
Second generation					
720	740	0	—	2.59	+
721	743	0	—	0.75	—
"	744	0	—	1.22	—
727	735	0	—	1.46	—
"	739	0	—		
742	736	0	—		
(777)	735	5	5		
"	739	0	—		
"	740	3	0		
(779)	743	0	—		
"	744	0	—		
"	735	4	0		
(898)	735	6	5		
"	739	0	—		
(697)	723	9	9		
"	726	9	8		
(699)	731	12	12		
"	732	12	12		
Young of females from 2d generation with males from 1st one					
932	930	0	—		
"	931	0	—		
933	934	6	0		
(923)	930	8	0		
"	931	6	4		

To decide the question whether in the experiments with maizefood after all still a lack of B-vitamin might interfere, we fed a number of rats with a maizefood I to which was added 5 % of marmite. The results are led down in table 7 from which we see, that not a single male of the second generation succeeded in propagating. The sterility of the males in this and in former experiments is not explainable by lack of B-vitamin.

Finally we added butterfat to a maize-peanut-codliveroil-food as it

TABLE 9. Ration XIIIa.

First generation			Weaned
Females	Males	Young	
804	838	5	5
821	840	8	6
824	838	6	0
	838	5	0
853	837	8	0
854	840	7	5
856	837	3	3
		Total 42	19 or 45 %

TABLE 10. Ration XIIIa.

Second generation			Weaned	Weight of testes	Spermatozoids
Female	Male	Young			
966	987	0	—	2.12	++
	(1003)	2[1]	0		
973	985	0	—	1.15	++
977	987	0	—		
	(991)	9	0		
981	986	9	2	2.27	++
982	986	0	—		
988	985	0	—		
	(1003)	4	4		
	Total 24		6 or 25 %		

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[1] Found in whomb after death.

seemed to me, that in the long run rats on artificial diets did better on butterfat than on codliveroil. The first generation showed normal fertility, in the second all males were sterile (table 8).

In our second communication we saw that male rats whose elders had lived on ration XIII (aceton extracted whole wheat meal, wheat starch, hardened fat, butterfat, casein, saltmixture and decitrated lemon juice) and after weaning got unextracted wheat meal, were infertile. We then investigated the case that the first generation got unextracted wheatmeal too. The results are colleced in table 9 and 10. In the first generation 45 % of the young were weaned; in the second only 25 %.

TABLE 11. Ration XIIIa with albumen e sangue.

First generation			Weaned	Weight of testes	Spermatozoids
Female	Male	Young			
818	835	8	3		
820	835	8	8		
843	817	3	0		
846	817	8	5		
992	994	7	7		
993	994	9	0		
	Total 43		23 or 53%		
Second generation.					
941	946	8	8	not investigated	
943	946	8	8		
945	953	0	—	2.35	+
945	(994)	0	—		
947	944	10	10		
949	944	4	0		
952	942	0	—	2.62	+
952	(995)	0	—		
954	953	0	—		
954	(994) [1]	0	—		
956	942	0	—		
956	(995) [1]	2	0		
	Total 32		26 or 81 %		

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[1] 994 and 995 had proved fertile with stockfed females.

The fertility of the females diminished as well. As albumen starvation or the use of an uncomplete albumen may be the cause of sterility, we tried an addition of albumen to the former diet. We chose albumen e sangue from BROCADES and STHEEMAN, that had prowed in other experiments to contain but very little vitamin A and B. We added 2 % of the dry feed. Table 11 gives the dates. Reproduction is a little hampered, as out of 12 matings only 5 succeeded, but of the 32 young born 26 were raised or 81 % that is little less than with stockrats.

We see thus a remarkable improvement of milksecretion. It is still dubious, wether this is due to the albumen itself or to impurities of vitamin character. For the albumen contains a little ether extractable matter. In rat 952 the autopsy discovered pathological alterations of uterus and ovaria that explain for the sterility.

From the discussed investigations we may conclude, that so for the production of fertilising spermatozoids as for normal gestation specifical fertility factors are indispensable in the nutrition, that may be compared with the other vitamins. For an efficient milksecretion also a specific factor is necessary, but it is not yet sure, that this is not an albumenfraction.

Physiological Laboratory Agricultural Academy
of Wageningen.

Anatomy. — *The eye-muscle nerves of petromyzonts, especially in their general morphological significance.* By J. L. ADDENS. (Central Dutch Institute for Brain-Research, Amsterdam.) (Communicated by Prof. B. BROUWER.)

(Communicated at the meeting of June 30, 1928).

The nuclei and roots of the eye-muscle nerves of petromyzonts, as compared with those of gnathostomes, show a number of differences, which, apart from their systematic value, give rise to some not unimportant conclusions of a general morphological nature. Hitherto these peculiarities were quite insufficiently known or erroneously interpreted. That there is not spoken here of a comparison between cyclostomes and gnathostomes, but between petromyzonts and gnathostomes, is due to the fact that in the other group of cyclostomes, the myxinoids, eye-muscle nerves as well as eye-muscles are totally lacking [1]).

For the present research were used a transverse, a longitudinal and a horizontal series through the brain of the adult Petromyzon fluviatilis, all three stained with EHRLICH's haematoxylin. Thickness of sections 18 μ.

To easier understand the following the reader is referred especially to figs. 1 and 2, representing longitudinal reconstructions of the motor nuclei and roots of Petromyzon fluviatilis and Selache maxima, resp., made after the method devised by KAPPERS. The chart of the shark Selache maxima was introduced to make possible the comparison with a less specialized lower vertebrate. Attention is also called to fig. 9, a part of fig. 1 two times enlarged, and with the intracerebral course of the roots indicated.

The remark must yet be made that in the ensuing speculations the cyclostomes, in accordance with the current view, are regarded as the most primitive craniotes extant, though in some respects strongly specialized, and in others reduced. In as far as the latter applies to the eyes, it is of special importance for the conclusions here arrived at regarding the eye-muscle nerves. TRETJAKOFF (1927), it is true, maintains that the eyes of petromyzonts are entirely normal, but, in my opinion, there are several characters suggesting that these organs once possessed a stronger development, among others, their smallness, the total lack of muscle-fibres within them, and the incompleteness of the tectum opticum. Moreover, the very rudimentary condition of the eyes of myxinoids gives additional weight to the surmise that also in petromyzonts, so closely resembling them in structure and life-habits, the visual powers are on the wane.

I now proceed to treat of the nuclei and roots of the three eye-muscle nerves in the sequence from before backwards, therefore starting with the oculomotor.

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[1]) The statement of some text-books that in myxinoids the eyes themselves are entirely absent, is not correct. The eyes and optic nerves are constantly present, although utterly rudimentary and perhaps no longer functional.

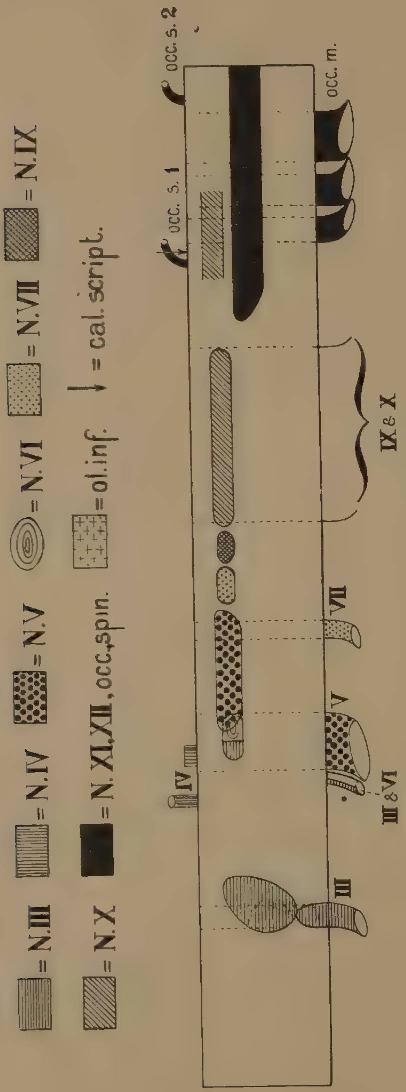


Fig. 1. *Petromyzon fluviatilis* L.

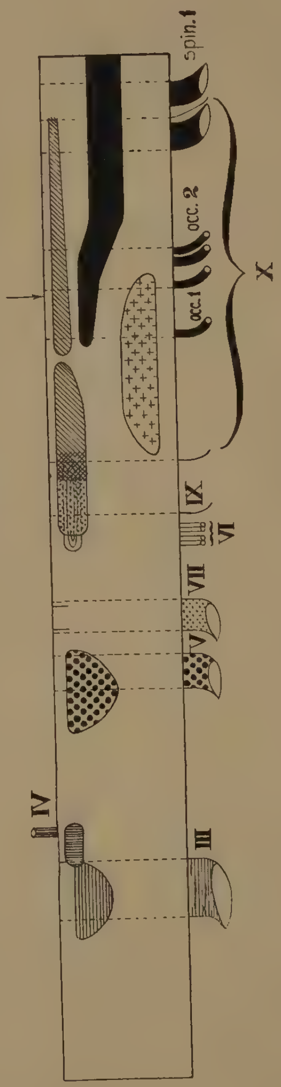


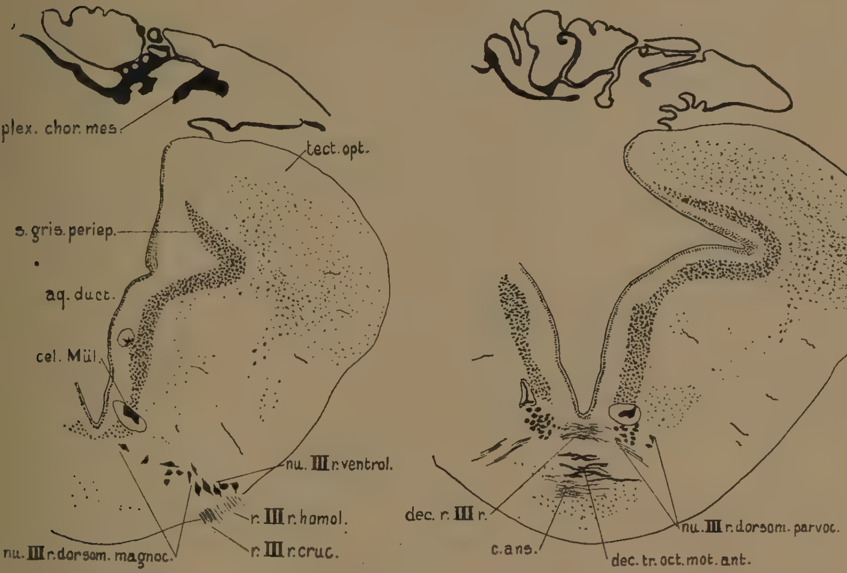
Fig. 2. *Selache maxima* (Gunn.)

The Oculomotor Nerve.

The oculomotor nucleus of cyclostomes, in contradistinction to that of gnathostomes, which always posses but one oculomotor nucleus, consists of two widely separated divisions, a rostral and a caudal. Of these two nuclei the rostral, which is situated at the same place as the single nucleus of the other vertebrates, is by far the more voluminous. The caudal nucleus, made up of but a small number of cells, lies far back at the level of the beginning of the trigeminal motor nucleus. The root arising from this caudal third nucleus, does not join that from the rostral or principal nucleus, but issues separated from the latter, in what manner we shall see later on.

Under the present heading I shall only deal with the rostral third nucleus and root, as the caudal ones are more aptly discussed together with the abducens.

In the rostral oculomotor nucleus two subdivisions are to be distinguished: a small ventrolateral and a large dorsomedial nucleus (figs. 1, 3 and 4).



Figs. 3 and 4. *Petromyzon fluviatilis*. Cross-sections through the mid-brain at the level of the front and hind parts, resp., of the rostral third nucleus. × 40. cel. Mül, Müllerian cell; c. ans., commissura ansulata; dec. r. III r., decussation of the rostral third root; nu. III. r. dorsom. magnoc., big-celled part of the dorsomedial rostral third nucleus; nu. III r. dorsom. parvoc., small-celled part of the dorsomedial rostral third nucleus; nu. III r. ventrol., ventrolateral rostral third nucleus; r. III r. cruc., crossed rostral third root; r. III r. homol., direct rostral third root.

The ventrolateral division lies close to the periphery of the mesencephalon, rather remote from the median plane. In the sagittal series it was noticed that a number of its cells, partly or even completely, lay a little outside the brain (fig. 5), but this could not be found either in the transverse or in the horizontal series. JOHNSTON (1902) observed also that in Lampetra wilderi some of the oculomotor cells project beyond the contour of the brain.

The cells of the ventrolateral nucleus are big, and stain intensely with haematoxylin, whereas those of the dorsomedial are smaller, and stain fainter. By these two differences the nuclei under discussion can always be clearly delimited, although they touch each other.

The dorsomedial nucleus can be subdivided into two zones, inasmuch as the cells of its hind dorsal part are smaller, and much more closely packed together than those of its foremost ventral part.

The fibres originating in this rostral oculomotor-complex, leave the brain in two bundles, viz., a slender compact bundle, which crosses, and a much broader bundle made up of loosely arranged fibers, which is uncrossed. These bundles are adjacent, the small crossing one occupying the medial position. The decussating fibres come exclusively from the dorsomedial nucleus, although many of the fibres from this nucleus show a direct course. The remainder of the homolateral fibres arises in the ventrolateral nucleus. The decussation of the oculumotor takes place at the hindmost level of the nucleus, and is best seen in horizontal sections (fig. 6).

The most conspicuous feature of the rostral third nucleus is the extremely ventral position of its lowest cells. In gnathostomes also this nucleus sometimes may stretch far ventrally, but never reaches the periphery of the brain, let alone that it extends beyond this boundary. Since, as remarked above, petromyzonts besides primitive also show secondary characters, we shall have to inquire, every time we encounter a deviation from the general vertebrate type, whether this is due to primitiveness or to secondary modification. Is, then, the ventral position of the third nucleus primitive or secondary? Before this question can be answered, we must

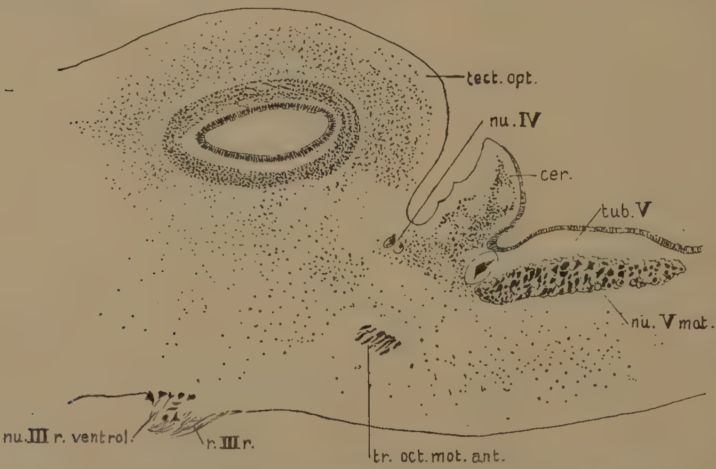


Fig. 5. *Petromyzon fluviatilis.* Parasagittal section through the mid-brain and the beginning of the oblongata. × 40. nu. III r. ventrol., ventrolateral rostral third nucleus; r. III r., rostral third root; tub. V, tuberculum trigemini.

consider what criteria in general are available regarding the primitiveness of the position of nuclei. For want of room, however, something can be said here only about the embryological criterion.

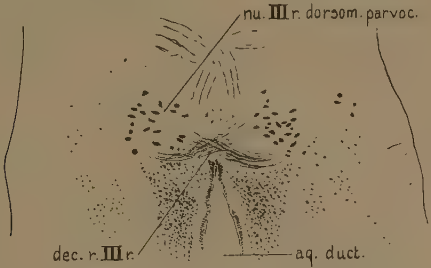


Fig. 6. *Petromyzon fluviatilis.* Horizontal section through the mid-brain at the level of the decussation of the rostral third root. × 40. nu.III r. dorsom. parvoc., small-celled part of the dorsomedial rostral third nucleus; dec. r. III r. decussation of the rostral third root.

According to the rule of recapitulation, motor nuclei which have shifted phylogenetically, will indicate their original location in the embryonic condition. The nearer, therefore, a nucleus lies in the adult to the place where it arose in the embryo, the more primitive its position. So migrations during ontogenesis are recapitulations of phylogenetic migrations. We must be here, of course, on our guard against cenogenesis: notwithstanding that a nucleus phylogenetically has shifted, it may already at its first appearance occupy its final place, or even arise at a quite different location from its original one.

The development of the motor nuclei of petromyzonts unfortunately is wholly unknown, but regarding higher vertebrates we are rather well-informed in this respect.

Bok (1915), especially, in his research on the development of the chick's brain has put the thesis that all motor cells originate as a single longitudinal column dorsal to the fasciculus longitudinalis medialis.

This thesis, however, is not tenable to its full extent, as is apparent from later work (Urra, 1922; Beccari, 1923; Tello, 1923), which my own researches confirm. In the first place, it must be remarked that the column in question is not continuous longitudinally, since several nuclei are isolated from the very first. A more serious objection is that the oculomotor and trigeminal nucleus do not arise medially, but laterally near the outer border of the brain, at least become visible as such here for the first time. Urra found even that in the youngest stages of the chick some cells of the third nucleus protrude outside the brain, just as described above for the adult Petromyzon. In the further course of development they migrate to their final dorsal position.

If this were a phenomenon of recapitulation, we would be bound to conclude that in primitive forms the place of the third nucleus was ventral, petromyzonts even nowadays showing this primitive location.

Notwithstanding that ontogeny points to primitiveness, I am inclined to regard the extremely ventral site of the third nucleus as secondary, and that for two reasons. First, because also the fifth nucleus, as mentioned above, arises laterally, and yet its original position is dorsal, as evidenced by the condition in lower vertebrates in general, inclusive of the petromyzonts themselves (see fig. 2 for the Selachii). So, phylogenetically, BOK's thesis may be right. Secondly, doubt is raised by the curious extra-cerebral situation of part of the cells.

I am of opinion, therefore, that it is more plausible to ascribe the character just discussed to a secondary shifting in a ventral direction, in consequence of the weakening of the optic impulses. The remark must be made, however, that the reflex pathways to the eye-muscle nuclei of petromyzonts are insufficiently known.

At this point it may be stated that nothing suggesting the presence of a nucleus of EDINGER-WESTPHAL was found. In this nucleus in all probability the greater part of the intrinsic eye-musculature (cf. BROUWER, 1918) is localized. Intrinsic eye-muscles as well as a ganglion ciliare are totally lacking in petromyzonts, as would be in keeping herewith.

Before proceeding to the caudal occulomotor nucleus and root, which are better discussed together with the abducens, I turn to the trochlear nerve.

The Trochlear Nerve.

The nucleus of the fourth nerve is very strangely located, inasmuch as it lies in the cerebellum, dorsal to the sulcus limitans therefore (figs. 5 and 7). Consequently it is not indicated within the chart, but above it.

The trochlear cells, which stain intensely with haematoxylin, are small and strongly elongated horizontally, i.e., in the direction of the axones going out from them. The axones leave their cells at the medial side, and cross above the ventricle. Having traversed the heterolateral nucleus, they collect to form the root, which emerges a fair distance in front of the nucleus in the usual dorsal position (fig. 8).

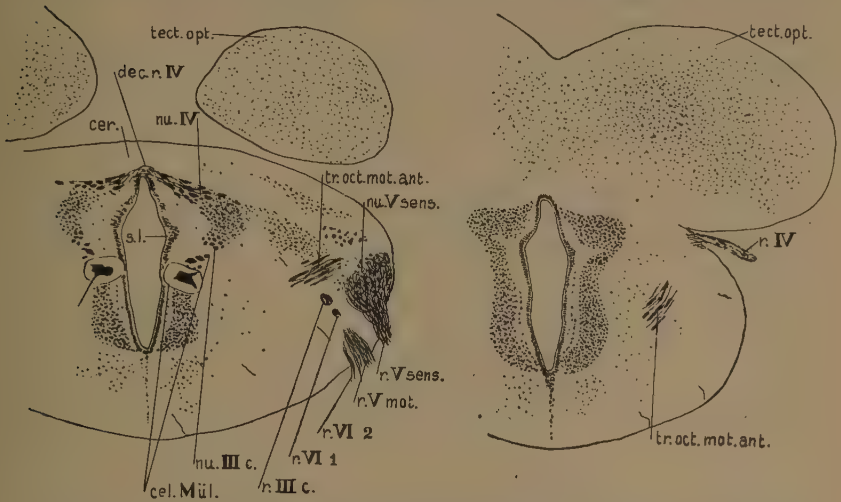
In two respects the location of the trochlear nucleus is noteworthy.

Firstly, because nowhere else is this nucleus encountered in the cerebellum, dorsal to the sulcus limitans, and secondly, because in no other group does it lie at so caudal a level.

Are these two features primitive or secondary?

Those investigators who have pronounced an opinion on the position of the trochlear nucleus of petromyzonts (TRETJAKOFF, 1909; KAPPERS, 1912, 1920) hold that both peculiarities are primitive. In this I can but partly agree with them.

It does not seem doubtful to me that the situation in the cerebellum is secondary. As intimated above, in petromyzonts nothing is known about the development of the motor nuclei, but in gnathostomes, as far as investigated, the trochlear nucleus always arises in the same way as the rest of the nuclei, i.e., in the basal plate, consequently beneath the sulcus limitans. It can be neurobiotactically explained, moreover, why this nucleus should have moved into the cerebellum, as set forth below.



Figs. 7 and 8. *Petromyzon fluviatilis.* Cross-sections through the oblongata and cerebellum at the level of the fourth nucleus and through the mid-brain at the level of the emergence of the fourth root, resp. × 40. nu. III c., caudal third nucleus; r. III c., caudal third root; r. VI 1, foremost sixth rootlet; r. VI 2, hindmost sixth rootlet; s. l., sulcus limitans.

The second peculiarity, the caudal position, must be accounted for inversily, for, as KAPPERS has convincingly pointed out, a caudal location of the trochlear nucleus is archaic. This topic may again be briefly discussed.

In the majority of vertebrates the fourth nucleus is situated immediately behind the third; only exceptionally are these nuclei separated by a more or less extensive interval, as especially in urodeles (RÖTHIG, 1913) where the trochlear nucleus lies almost as caudally as in petromyzonts. This nucleus, in all cases in which its ontogeny was investigated [1]), was found to arise a little caudal to the third nucleus, shifting forwards towards the latter as development proceeds. The assumption that a caudal position of the trochlear nucleus is primitive is further substantiated by the fact, discovered by TSUCHIDA (1906), and confirmed by KAPPERS (1912) and VAN VALKENBURG (1912), that in man frequently a part of the nucleus lies separately at a more caudal level (nucleus trochlearis posterior).

Although all this speaks strongly in favour of the primitiveness of a caudal situation of the trochlear nucleus, yet it is very questionable if this feature, in its full extent, may be attributed to primitiveness, as the embryonic interval between the third and fourth nucleus is always quite small, and in no way so extensive as in adult petromyzonts and urodeles. The caudal position of the fourth nucleus in these forms will, therefore, probably be due partly to backward shifting.

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[1]) For full literature, as for all problems touched upon in this preliminary communication, the reader is referred to a forthcoming more complete account.

The Oculomotor-abducens Nerve.

I now proceed to the common treatment of the caudal oculomotor and abducens nerves, which henceforward, as forming a single nerve-stem, will be designated as the oculomotor-abducens.

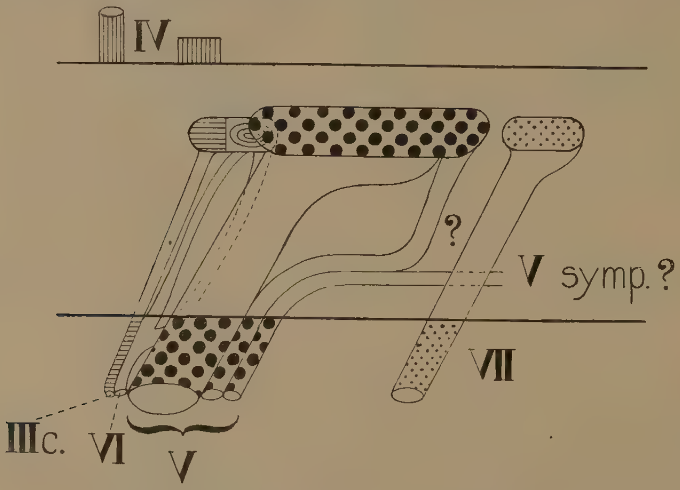

Fig. 9. *Petromyzon fluviatilis.* A part of fig. 1 two times enlarged with the course of the roots indicated. IIIc., caudal third root.

The information available in literature about this nerve, or better said, about the abducens, since it has not been hitherto recognised that oculomotor elements are present in it, is utterly confused, and for the greater part wrong. A knowledge of the anatomical details involved is gained most easily by a survey of the scheme of fig. 9.

Ventromedially and adjacent to the rostral end of the trigeminal nucleus, which consists of big cells, there is found a column of much smaller cells (fig. 10). In its greater part, however, this column extends in front of the fifth nucleus (fig. 7). By the size of the composing cells it can be divided into two parts, the foremost and shorter part being made up of somewhat smaller elements than the hindmost and longer.

To say it at once, without giving reasons for the present, the foremost component of the column in question is the caudal oculomotor nucleus spoken of above, the hindmost the abducens nucleus.

From the caudal oculomotor nucleus one rootlet departs, from the abducens nucleus two. The foremost of the two sixth rootlets, during its intracerebral course, is situated close to the oculomotor rootlet, the hindmost, however, more separately. The third and foremost sixth rootlets, on leaving the brain, unite, and a little outside the brain the hindmost sixth rootlet joins them. Thus these three fibre-bundles form one stem, the oculomotor-abducens, the abducens of my predecessors. It runs but for a short distance independently, and soon unites with the motor trigeminal. Its further course could not be followed in the series at my disposal, since they only contained the brain with the proximal nerve-stumps. Regarding this, however, we are informed long ago by the researches of Paul Fürbringer (1875), a brother of the renowned comparative anatomist Max Fürbringer. In the orbit the oculomotor-abducens, as appears from P. Fürbringer's description of Petromyzon marinus, has left the trigeminus, and again has become independent.

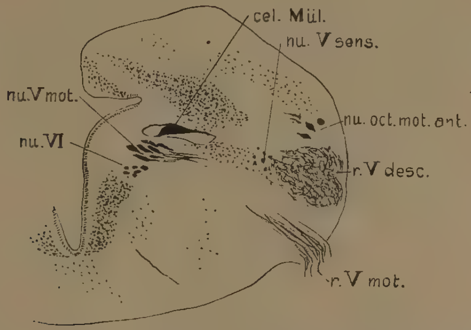

Fig. 10. *Petromyzon fluviatilis.* Cross section through the oblongata at the level of the sixth nucleus × 40.

Here it divides into two branches, one of which goes to the rectus posterior s. externus, the usual abducens muscle, whilst the other innervates the rectus inferior, which in all other vertebrates is supplied by the oculomotor.

The correctness of P. Fürbringer's observations is not to be doubted. M. Fürbringer (1897) and Corning (1902) confirmed them for the same species (marinus). I, too, could by dissection of a large specimen of Petromyzon marinus ascertain this mode of innervation, and Nishi (1922) found the same in Entosphenus japonicus.

Petromyzon fluviatilis behaves somewhat differently in this respect, according to the recent papers of Tretjakoff (1926, 1927). Though here, too, the rectus inferior is mainly supplied by the oculomotor-abducens, it still gets a small twig from the rostral oculomotor [1]).

It will be clear by now, how I was led to interpret the so-called abducens -

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[1]) Tretjakoff, of course, speaks simply of oculomotor and abducens.

as an oculomotor-abducens. This conception was, quite naturally, arrived at by combining the fact that two parts could be seen in the nucleus of this nerve with the aberrant territory of its innervation.

According to this supposition, a part of the oculomotor-cells, and just the one innervating the rectus inferior, has migrated backwards, and joined the abducens nucleus. In Petromyzon marinus this process has been complete, in Petromyzon fluviatilis some of the rectus inferior cells have remained in their old centre. Thence the twig found by TRETJAKOFF.

If it might be objected that this is too extensive a migration, it can be immediately replied that the abducens nucleus, about whose identity no doubt is possible, must have shifted forewards over at least the same distance, as a comparison between the charts of Petromyzon and Selache shows. In the latter form the sixth nucleus approximately exhibits its original position, as will be argued presently. Likewise the sixth root here emerges at its original level.

Of course, the hypothesis just put forward only can be regarded as proved, when it is established that during development cells from the third nucleus migrate backwards, and join the sixth nucleus ; but, as remarked, nothing is known about the ontogeny of the motor nuclei of petromyzonts.

It may now be mentioned briefly what was known heretofore about the so-called abducens.

P. FÜRBRINGER (1875) described the course of the nerve in the orbit, and discovered the remarkable innervation, whilst AHLBORN (1883), for the first time, observed its emergence close in front of the trigeminus. Thus, it was the central relations which hitherto were not elucidated. According to AHLBORN, the root springs from the foremost part of the trigeminal nucleus, JOHNSTON (1902, 1905) maintains that the sixth nucleus occupies its usual place, and TRETJAKOFF wholly arbitrarily assumes that it lies in the hindmost part of the trigeminal nucleus.

The only author that saw the oculomotor-abducens column, without noticing, however, two parts in it, was TRETJAKOFF, according to whom the fibres arising from it run forwards, and emerge with the rostral oculomotor. Consequently he considers this cell-group in its entirety as an oculomotor nucleus (his dorsal oculomotor nucleus). I could find no trace of such fibres.

The numerous efforts hitherto made to explain why the so-called abducens of petromyzonts in addition to its own accustomed muscle, the rectus posterior, innervates the rectus inferior, cannot all be reviewed here.

P. FÜRBRINGER and others have tried to solve this enigma by supposing that some of the eye-muscles of petromyzonts are not homologous with their namesakes in the rest of vertebrates.

P. FÜRBRINGER, for instance, assumes that the rectus posterior and inferior of petromyzonts have united in gnathostomes to form one muscle : the rectus posterior of the latter. Thus in them but one muscle would be supplied by the abducens. The rectus anterior of petromyzonts at the same time would have split up into the rectus anterior and inferior of gnathostomes, the latter supplanting the homonymous muscle of petromyzonts.

PLATE, in his recent text-book of General Zoölogy (1922, 1924), has put forward a different explanation, though he, too, regards the innervation of the rectus inferior by the abducens as a primitive cyclostome feature. In gnathostomes the muscle in question would have shifted forwards into the territory of the oculomotor, where this nerve would have taken over the task of the abducens. But, although in petromyzonts the rectus inferior is situated somewhat further back than in gnathostomes, this difference is much too slight to make acceptable such an explanation.

PLATE's hypothesis rests on the principle that changes of innervation during phylogenesis are possible, a phenomenon designated by the name of secondary, collateral or heterotopic innervation, which last term, as the most characteristic, will be used by me throughout.

Finally there is an explanation by MOŽEJKO (in "BRONN's Klassen und Ordnungen"), very intricate and far-fetched, but nevertheless accepted by TRETJAKOFF, which cannot be dealt with here.

CORNING (1902) alone has surmised, in what direction at least, the solution of the problem lay, viz., in supposing that intracranially oculomotor fibres have joined the abducens.

To the phenomenon that within the central nervous system a part of one nerve unites with another, and emerges with it, I should like to give the name of *central anastomosis*. Nerves, therefore, are not fixed units, not even the so strongly individualized cranial nerves. An investigation, extending over the motor nuclei and roots of all the vertebrate groups, has revealed to me other instances of this phenomenon, though it is not frequent.

In two respects this principle of central anastomosis is of importance.

In the first place, from the standpoint of pure morphology, whereby here is understood the mere establishing of homologies. FÜRBRINGER, and following him the school of GEGENBAUR in general, was of opinion that during phylogeny the innervation of muscles never changed, and so would be an infallible guide in tracing the homology of the latter. This view rested on the now obsolete HENSEN doctrine that motor cell and muscle-fibre are connected from their earliest ontogenetic appearance. Although all comparative anatomists will be agreed as to the correctness of FÜRBRINGER's rule in the overwhelming majority of cases, many among them, as apparent from the foregoing, ascribe a more or less important rôle to secondary or heterotopic innervation, as especially EDGEWORTH (1911), who enumerates a whole series of examples. E. HUBER [1]), on the contrary, has even most recently defended the absolute validity of FÜRBRINGER's doctrine. In a series of studies on the facialis musculature he, by electrically stimulating or cutting the facialis, reached the result that the alleged additional supply of these muscles by the plexus cervicalis or trigeminus is due to peripheral anastomoses.

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[1]) See for E. HUBER's writings on this topic the list of literature in HUBER and HUGHSON (1926).

My own attitude in this matter is conciliatory.

On the one hand I am of opinion that EDGEWORTH goes too far. Not to speak of erroneous observation, I believe that a number of cases of heterotopic innervation is but seeming, and once will be eliminated by the demonstration of peripheral, or perhaps central anastomoses, as done in the present research for the rectus inferior of petromyzonts. Moreover, there is another complication by which a false appearance of heterotopic innervation can be raised, viz., that fibres in the course of evolution transferred their place of exit from ventral to dorsal. To this I presently shall return.

On the other hand, however, I cannot entirely agree with E. HUBER. Without doubt, there are cases in which in the course of time the nerve supply of muscles has undergone alteration. The most striking and certain example, known to me, is brought out by a comparison of the innervation of the tail of urodeles and reptiles with that of mammals (V. SCHUMACHER, 1909). The spinal cord in the first-named animals continues backwards to the tip of the tail, giving off in each segment a pair of caudal nerves. In mammals, on the contrary, the caudal part of the spinal cord is reduced, and along with it the last tail-nerves, so that the number of caudal vertebrae is from three to six times more numerous than that of the tail-nerves. In the embryo these tail nerves which have vanished in the adult, are still laid down, but soon they disappear. The innervation of the muscles of the last tail-segments, consequently, must have been taken over by more rostral tail-nerves.

Thus having considered the significance the principle of central anastomosis may have in the controverse regarding the phylogenetic constancy of innervation, I turn to the second theoretical point for which the phenomenon is of importance. KAPPERS has put forward the law that nerve-fibres which conduct impulses at the same time or shortly one after the other in the same direction, tend to group together in bundles. This he called *fasciculation*. In the special case of motor fibres the simultaneousness of excitation would be brought about by the circumstance that the uniting fibres innervate closely collaborating muscles. As we shall see presently, however, the simultaneousness may have another cause.

Is, then, the union of oculomotor and abducens fibres described above, a case of fasciculation?

This would not seem quite impossible, although in no event is it a striking illustration of the principle. The caudal oculomotor goes to the rectus inferior, the abducens to the rectus posterior. P. FÜRBRINGER (1875) as well as TRETJAKOFF (1926) states that in Petromyzon the eye-ball is turned backwards by the rectus posterior, and downwards and at the same time a little *backwards* by the rectus inferior. Thus, these two muscles may collaborate to some extent ; if here, however, stronger than in gnathostomes in general I should not venture to say for the moment. In a forthcoming more complete account I hope to be in a position to say something more regarding this point.

It is apparent from the foregoing that the positions of both the abducens and of the caudal oculomotor nucleus of Petromyzon are held by me to be secondary. For the abducens nucleus this is easily demonstrated. Mainly by the researches of STREETER (1908) in man, but also by those of BOK (1915) on the chick's development, the nucleus in question is known to arise in the position characteristic for adult selachians (see fig. 2 for Selache). It follows, furthermore, from the development of the muscle localized in the sixth nucleus, that the latter is rather to be expected on the level of the facialis than on that of the trigeminal nucleus, this muscle, the rectus posterior, being formed by the upper part of the hyoid myomere, the myomere of the facialis.

The location of the caudal oculomotor nucleus, likewise, is not a primitive character. As an argument can be adduced that in those gnathostomes where the ontogeny of the third nucleus was studied, never did some of its cells arise more caudally and apart from the rest. It is possible, moreover, to ascribe this caudal position to neurobiotactic influences, and to the same that have brought about the location of the abducens and trochlear nuclei.

Which neurobiotactic factors, then, have been operative here?

The close spatial relationship of all three nuclei to the cerebellum provides us with a key. The caudal third nucleus lies just beneath this part of the brain, the sixth immediately behind, and the fourth even within it. The afferent pathways to the cerebellum of petromyzonts are almost exclusively of octavolateral nature. So it is justified to ascribe the position of the three nuclei at the same longitudinal level to the neurobiotactic influence of static and lateralis stimuli, which prevailed over the agency of the optic system.

The second peculiarity of the oculomotor-abducens calling for comment, is the lateral emergence of its root. As aforesaid the latter leaves the brain in as lateral a position as the motor trigeminus. In gnathostomes, without exception, the oculomotor and the abducens emerge ventrally, which is deemed a typical feature of so-called somato-motor nerves. Probably also this peculiarity is secondary, as it is not at all impossible to account for it by fasciculation, on the basis of the specialized manner of life of these animals.

This explanation, however, can only be given with great reserve. A little digression on the feeding habits of petromyzonts is necessary to this end.

The petromyzonts are predatory. They attack fishes, even very big ones. With its oral funnel the lamprey fixes itself to its prey, and by the rasping movements of the so-called tongue, which is provided with teeth, it reduces the tissues of its victim to a fine pulp, which is then swallowed.

The so-called tongue is supplied by the trigeminus, whence appears that this organ has nothing to do with a genuine tongue. While the lamprey is feeding, the prey, roused by pain, will carry with it the aggressor in convulsive movements. So the eye-muscle nerves, which mainly receive static and lateralis impulses, will be stimulated simultaneously with the trigeminus, which moves the tongue.

If the union of oculomotor-abducens and trigeminus actually is a consequence of simultaneousness of stimulation, then, in this case, as alluded to above, the simultaneousness may not be ascribed to the contraction of closely collaborating muscles, since it cannot be seen how eye-movements could be of any avail to the lamprey while feeding. The contraction of the one group of muscles merely entails that of the other. But, as stated, the above explanation is only given with all due reserve. It is possible, after all, that any weak nerve, for some reason or other coming into the vicinity of a strong nerve-stem, may unite with it.

However this may be, the lateral emergence of the oculomotor-abducens is a remarkable fact, and of great interest from the point of view of pure morphology also. It stands not alone as an instance of the phenomenon of a nerve transferring its place of exit from ventral to lateral or dorsal. The spinal accessory and a part of the vagus have arisen in like manner, as was brought to light during my researches on the phylogeny of the motor cranial nerves of vertebrates, already mentioned above.

Something may yet be said here on the origin of the accessory. This nerve until now was, following GEGENBAUR and FÜRBRINGER, generally interpreted as a branch of the vagus which had gained during evolution a more or less pronounced independence. The musculature supplied by it, the trapezius musculature, in keeping herewith is regarded as a derivative of the branchial muscles, as seemed proved by the embryological researches of EDGEWORTH (1911). It was, however, found out by me that the accessory is nothing else but fibres of one or more of the foremost spinal nerves which instead of a ventral have assumed a dorsal exit, and have joined the vagus. So there is ground for presuming that EDGEWORTH was mistaken in his researches on the development of the trapezius. In justification of this surmise I can appeal to a paper of VÖLKER (1908) on the development of the occipital region of Larus ridibundus, which clearly describes and figures that the trapezius is formed by the fusion of split-off portions of the occipital myotomes.

For want of room it cannot be further pointed out what theoretical significance the phenomenon of nerves taking on a dorsal exit instead of a ventral has, for instance, for the so-called "four-root" or American theory about the division of the nervous system. It may only be mentioned that in this manner, too, a fallacious appearance of heterotopic innervation can be raised, inasmuch as nerves which really are identical, may seem quite different by their changed mode of exit. The translocation in question probably is caused by fasciculation, but about this no more can be said for the moment.

Let us now summarize what was described or argued in the above.

For the first time a plausible explanation was given of the strange fact, discovered by P. FÜRBRINGER in 1875, that in petromyzonts the abducens not only innervates the rectus posterior, but also the rectus inferior, viz., by making it probable that the abducens of these animals is not a pure abducens, an oculomotor bundle having become associated with it within the brain. To this and similar phenomena the name of *central anastomosis* was given.

Furthermore, an endeavour was made to explain the deviations from the general vertebrate type which petromyzonts exhibit in the structure of their eye-muscles, on the basis of secondary modification, mainly as the effect of a diminution of their visual powers. The characters thus interpreted, are no less than six in number:

1. The extremely ventral or even extracerebral situation of the lowest rostral oculomotor cells.
2. The presence of a caudal third nucleus, united with the sixth.
3. The location of the fourth nucleus in the cerebellum.
4. The extremely rostral position of the sixth nucleus.
5. The union of the caudal third and sixth roots to form an oculomotor-abducens, emerging close in front of the trigeminus.
6. The lateral emergence of the oculomotor-abducens.

The extremely caudal position of the fourth nucleus, moreover, is probably but partly a consequence of primitiveness.

It was possible to attribute most of these peculiarities to neurobiotaxis, and two of them perhaps to fasciculation. It may be insisted how a comparison of the charts of Petromyzon and Selache not only strikingly demonstrates the existence of the shifting phenomenon, but also conveys an impression over what considerable distances the migrations of the nuclei may extend.

Finally, attention is directed towards the two principles of pure morphological nature arrived at in this paper, partly on the basis of the above facts. First, that even the so strongly individualized cranial nerves are not fixed units, which, apart from the doctrine of neurobiotaxis and fasciculation, is of importance in the problem of heterotopic innervation, and secondly, that the place of exit of roots may change from ventral to dorsal. It could only be cursorily touched upon what significance the latter phenomenon has in various morphological problems.

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Mathematics. — *On* Hilbert's *Function, Series of Composition of Ideals and a generalisation of the Theorem of* Bezout. By Bartel L. van der Waerden. (Communicated by Prof. R. Weitzenböck.)

(Communicated at the meeting of September 24, 1927).

By the aid of his "chains of syzygies" Hilbert 1) has shown that the number of linear equations which the coefficients of a polynomial of the degree ϱ must satisfy in order to belong to a given H-ideal ("Formenmodul") may be represented by a whole rational function of ϱ, the "characteristic function", if ϱ has a sufficiently high value. Lasker 2) has proved this result again in a simpler way, and has shown that the degree of the characteristic function is connected with the dimension of the ideal. If we write the function as a sum of binomial coefficients $\binom{\varrho}{k}$ multiplied by constants c_k, these c are integer numbers that are characteristic for the ideal.

Another system of such numbers, among which the *exponent* and the *length* are the most important ones, can be defined not only for the polynomial domains but also for arbitrary primary ideals in arbitrary „ring" domains, where Hilbert's basis theorem holds good. These numbers are exclusively dependent on the domain of restclasses, not on the initial domain. In what follows we shall only investigate the "length", i.e. the length of a series of composition of primary ideals which terminates in a given primary ideal.

This length is closely connected with the characteristic function. In the simplest case: when the H-ideal is primary with only one zero in projective space, the characteristic function is a constant and equal to the length. The coefficient of highest index of the characteristic function, for wich we use the name "degree" of the ideal, is always equal to the sum of the degrees of the primary components of the largest dimension and the degree of a primary ideal is equal to the length multiplied by the degree of the corresponding prime ideal, which latter is again equal to the degree of the manifold of zeros of this prime ideal in projective space.

The latter fact gives importance to the theory of the characteristic function for geometry: the theory makes it possible to establish a relation

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1) D. Hilbert, Ueber die Theorie der algebraïschen Formen, Math. Ann. **36**, (1890) p. 473.

2) E. Lasker, Zur Theorie der Moduln und Ideale, Math. Ann. **60**, (1905) p. 20; Cf. also A. Ostrowski, Abh. Math. Sem. Hamburg, **1**, (1922) p. 281 and F. S. Macaulay, Proc. London Mathem. Soc. **26**, (1926) p. 531.

between the degree of an algebraic manifold and the degree of the intersection of this manifold with a spread $f = 0$. This relation is a generalisation of the theorem of BEZOUT.[1])

All these things are indicated partly in the paper of LASKER, partly in a report of an address in Göttingen by E. NOETHER[2]); however, they have not yet been investigated with the necessary precision. It seemed, therefore, desirable to treat the whole connected complex again with the most modern and most simple methods (especially without chains of syzygies and without the theory of elimination).

In what follows we shall at once treat the more general case in which the functions under consideration are homogeneous in more than one system of variables. This gives nothing essentially new, nor is it more difficult, and the results have importance for certain geometrical applications, which LASKER has already pointed out. In order not to be drowned in indices I shall introduce 2 systems of variables only; it will at once be clear how the results read for more than 2 systems of variables.

I shall only suppose the principal notions exposed in my "Nullstellentheorie der Polynomideale" [3]) to be known.

After an introductory part I, which contains only formal trivialities, part II gives a theory of the characteristic function entirely based on LASKER's method of argumentation. III deals in a general way with series of composition for primary ideals, independently of the preceding. For this part a manuscript of E. NOETHER has been gratefully made use of. IV connects the different ways of investigation and V gives the geometrical application.

I. *Homogeneous ideals and multifold-projective spaces.*

§ 1. Let Γ be a field and $x_0, \dots, x_n, y_0, \dots, y_m$ variables. By *forms* we shall understand such polynomials in $R = \Gamma[x_0, \dots, y_m]$ as are homogeneous in $x_0, \dots, x_n$ as well as in $y_0, \dots, y_m$. A *homogeneous ideal* or *H-ideal* in R is an ideal that together with any polynomial f always contains all homogeneous additive component parts of f. Evidently a homogeneous ideal has a homogeneous basis, for the polynomials of an arbitrary basis can be decomposed into homogeneous component parts. Conversely any ideal for which a basis consisting of forms can be found, is homogeneous.

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[1]) This generalisation differs from the one which I published in Math. Ann. **99**, p. 497: there we have to do with the far more difficult case that an M_r and an M_{n-r} in projective space P_n intersect, whereas in what follows here the intersection of an M_r and an M_{n-1} is investigated. This separation is due to methodical reasons: the simplest case of the intersection of a curve M_1 with a spread M_{n-1} may be treated with the characteristic function only; by the aid of the notion length the theory can easily be extended to M_r en M_{n-1}; but for the other generalisation M_r and M_{n-r} these two notions are certainly insufficient: in this case we need besides the characteristic function also the general notion of multiplicity as has been indicated in the paper in question in the Math. Ann.

[2]) Jahresberichte Deutsche Mathematiker-Vereinigung **34**, (1925) p. 101.

[3]) Math. Ann. **96**, (1926) p. 183, §§ 1—5.

§ 2. By a *point of Cartesian space* C_{n+m} we understand a system of elements $\{\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m\}$ of Γ or of a field containing Γ. By a point $x = \{\xi_0, \dots, \xi_n; \eta_0, \dots, \eta_m\}$ of the *twofold-projective space* $P_{n.m}$ we understand a class of systems of elements $\{\lambda\xi_0, \dots, \lambda\xi_n; \mu\xi_0, \dots, \mu\xi_n\}$; where $\xi_0, \dots, \xi_n, \eta_0, \dots, \eta_m$ are fixed elements of Γ or of a field containing Γ, so that not all $\xi_i = 0$ and not all $\eta_i = 0$, and where λ and μ describe all elements of Γ or of the larger field in question. If it is necessary to indicate more closely which larger field Ω we have in view, we write $C_{n+m}(\Omega)$ resp. $P_{n.m}(\Omega)$. The $(n+1)+(m+1)$ elements ξ_i, η_k are called *coordinates* of the point. The elements $\lambda\xi_i, \mu\eta_k$ $(\lambda \neq 0, \mu \neq 0)$ may also be used as coordinates of the same point. The points $\{\xi, \eta\}$ of $P_{n.m}$ are evidently in a one-one correspondence with the pairs of points $\{\xi\}, \{\eta\}$ of two single projective spaces P_n, P_m.

If $\xi_0 \neq 0, \eta_0 \neq 0$, we can associate a point

$$\overline{X} = \left\{\frac{\xi_1}{\xi_0}, \dots, \frac{\xi_n}{\xi_0}; \frac{\eta_1}{\eta_0}, \dots, \frac{\eta_m}{\eta_0}\right\}$$

of C_{n+m} to the point $X = \{\xi, \eta\}$ of $P_{n.m}$. In this way the points X of $P_{n.m}$ for which $\xi_0 \neq 0, \eta_0 \neq 0$, are in a one-one correspondence with the points $\overline{X}$ of C_{n+m}. The elements $\frac{\xi_i}{\xi_0}, \frac{\eta_i}{\eta_0}$ are called *non-homogeneous coordinates* of the point X. If a finite number of points is given, we can always attain by a linear transformation of coordinates that we have for all these points at the same time $\xi_0 \eta_0 \neq 0$. This is at once possible if the field Γ has an infinite number of elements; in the other case the field must first be extended, e.g. by adjunction of variables.

§ 3. By *zeros* of an ideal M in R we understand such systems of values $\{\xi_0, \dots, \xi_n, \eta_0, \dots, \eta_m\}$ for which all polynomial, of the ideal become $= 0$. The elements ξ_i, η_i may be taken from an arbitrary domain containing Γ. All zeros of an ideal constitute together the *manifold* of the ideal in C_{n+m+2}.

Let M be an H-ideal in R. We suppose that M has zeros where not all $\xi_i = 0$ and not all $\eta_i = 0$. If $\{\xi, \eta\}$ is such a zero, all systems of elements $\{\lambda\xi, \mu\eta\}$ are zeros and these may be united to a "*class of zeros*", which is a point of the projective space $P_{n.m}$. The manifold of M may, therefore, also be considered as an algebraic manifold in $P_{n.m}$.

We shall now show that between the H-ideals in $R = \Gamma[x_0, \dots, x_n, y_0, \dots, y_m]$ and the ideals in $\overline{R} = \Gamma[x_1, \dots, x_n, y_1, \dots, y_m]$ there exists a correspondence analogous to that between the points X of $P_{n.m}$ and the points $\overline{X}$ of C_{n+m} (§ 2).

§ 4. If in a polynomial $f(x, y)$ of R we put $x_0 = 1, y_0 = 1$, we get a polynomial $\overline{f}$ of $\overline{R}$. In this way we get any polynomial $\overline{f}$ of $\overline{R}$ at least once. Sums $f + g$ and products $f . g$ pass into sums $\overline{f} + \overline{g}$ and products $\overline{f} . \overline{g}$. (Homomorphism or meroedric isomorphism). Hence an ideal M in R transforms into an ideal $\overline{M}$ in $\overline{R}$.

If as a special case for M we choose an H-ideal, we need not take all polynomials f of M for the determination of the polynomials $\overline{f}$ of $\overline{M}$, but we can restrict ourselves to the forms of f. For we can split up any polynomial f of M into homogeneous component parts which belong likewise to M, and we can multiply these component parts by such factors $x_0^\rho y_0^\sigma$ that they get the same degree. In this case the sum is a form of M, and for $x_0 = 1$ it gives the same as the original polynomial.

§ 5. If conversely an ideal $\overline{M}$ of $\overline{R}$ is given, all forms of R that through the substitution $x_0 = 1$ are transformed into polynomials of $\overline{M}$, generate an H-ideal M_0, which, apparently, through the substitution $x_0 = 1$, $y_0 = 1$ again transforms into the ideal $\overline{M}$. The H-ideal M_0 constructed in this way, is called *the H-ideal equivalent to $\overline{M}$.*

§ 6. The following formal relations are easily verified:

1. If $M = f_1, \ldots, f_r)$, then $\overline{M} = (\overline{f}_1, \ldots, \overline{f}_r)$.
2. $\overline{(M, N)} = \overline{M}, \overline{N})$ (Consequence of 1).
3. $\overline{M . N} = \overline{M} . \overline{N}$ (Consequence of 1).
4. If M is primary and P the corresponding prime ideal, then $\overline{M}$ is primary, and $\overline{P}$ is the corresponding prime ideal. (If M is prime, $\overline{M}$ is also prime).
5. If $\overline{M}$ is primary and $\overline{P}$ the corresponding prime ideal, M_0 is also primary and P_0 the corresponding prime ideal. (If $\overline{M}$ is prime, M_0 is also prime).

§ 7. Different H-ideals M can give the same ideal $\overline{M}$; among these H-ideals the H-ideal M_0 that is equivalent to $\overline{M}$ plays a special part. M_0 can be defined directly by means of M:

THEOREM 1. *M_0 is the aggregate of all the polynomials f that satisfy a congruence of the form*

$$x_0^\rho y_0^\sigma f \equiv 0\ (M)$$

PROOF. As M_0 is an H-ideal, it is sufficient to prove the theorem for the forms in M_0. The forms in M_0 are those forms f_0 which by the substitution $x_0 = 1$ pass into polynomials $\overline{f}$ of $\overline{M}$, and these polynomials arise in their turn by the substitution $x_0 = 1$ from forms

f of M. Now if two forms f_0 and f by the substitution $x_0 = 1$, $y_0 = 1$ lead to the same polynomial $\bar{f}$, they only differ from each other by factors x_0 and y_0 and conversely. From this follows what was to be proved.

As a consequence of this theorem if in all zeros $\{\xi, \eta\}$ of the ideal M we always have either $\xi_0 = 0$ or $\eta_0 = 0$, then M_0 is the unity-ideal. For if the polynomial $x_0 y_0$ becomes equal to zero in all zeros of M, we have, according to HILBERT's theorem of zeros,

$$x_0^\lambda \, y_0^\lambda \, . \, 1 \equiv 0 \, (M)$$

and consequently 1 belongs to M_0.

§ 8. THEOREM 2. *If M is given as an intersection* (*L. C. M.*) *of primary ideals:*

$$M = [Q_1, \ldots, Q_r]$$

and if among the ideals $Q_1, \ldots, Q_r$ only $Q_1, \ldots, Q_s$ have the property that in all their zeros $\xi_0 \eta_0 = 0$, we have

$$M_0 = [Q_{s+1}, \ldots, Q_r] \qquad (\textit{resp. } M_0 = (1) \textit{ for } s = r).$$

PROOF. A polynomial f belongs only then to M_0 if (for a sufficiently large ϱ and σ) $x_0^\varrho y_0^\sigma f$ belongs to M, i.e. to all Q_i $(i = 1, \ldots, r)$. For $i = 1, \ldots, s$ the condition $x_0^\varrho y_0^\sigma f \equiv 0 \, (Q_i)$ for sufficiently large ϱ, σ is no condition at all for f, for a power $x_0^\lambda y_0^\lambda$ lies ipso facto in Q_i (see final remark in § 7). For $i = s+1, \ldots, r$ $x_0^\varrho y_0^\sigma f \equiv 0 \, (Q_i)$ is equivalent to $f \equiv 0 \, (Q_i)$, as $x_0^\varrho y_0^\sigma$ does not contain the manifold of Q. Hence $x_0^\varrho y_0^\sigma f \equiv 0 \, (M)$ is equivalent to $f \equiv 0 \, (Q_i)$ $(i = s+1, \ldots, r)$, q. e. d.

§ 9. We now choose among the ideals $Q_1, \ldots, Q_r$ those that have only zeros $\{0, \ldots, 0;\ \eta_0, \ldots, \eta_m\}$ or $\{\xi_0, \ldots, \xi_m;\ 0, \ldots, 0\}$, in other words that have no zeros at all in projective space $P_{n,m}$. We shall call such ideals *projectively irrelevant*. For each of the other Q_i we can find a zero where at least one $\xi_i \neq 0$ and at least one $\eta_k \neq 0$. By means of a linear transformation (if necessary after extension of the field Γ) we can ensure that in this finite number of points at the same time $\xi_0 \neq 0$ and $\eta_0 \neq 0$ hold. In what follows we shall always suppose this transformation to have been applied beforehand. By so doing we ensure that the projectively irrelevant components of M are the only ones whose zeros are contained in $\xi_0 \eta_0 = 0$. We can now formulate theorem 2 in the following way:

If M is given as an intersection of primary ideals, we get M_0 by omitting the projectively irrelevant components.

§ 10. If $X = \{\lambda \xi, \mu \eta\}$ is a class of zeros of M in $P_{n,m}$, and if $\xi_0 \neq 0$, $\eta_0 \neq 0$, we can suppose $\xi_0 = 1$, $\eta_0 = 1$. In this case $\overline{X} = \{\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m\}$ is a zero of $\overline{M}$ in C_{n+m}.

If, conversely, $\overline{X}$ is a zero of $\overline{M}$, K is a class of zeros of M. We get, therefore, the manifold of $\overline{M}$ in C_{n+m} from that of M in $P_{n,m}$, by finding the points $\overline{X}$ in C_{n+m} corresponding to all points X of the latter manifold where $\xi_0 \neq 0$ and $\eta_0 \neq 0$.

§ 11. If, especially, $M = P$ is a prime ideal that is not projectively irrelevant, we have, according to the theorem of § 9, $P_0 = P$. Further we have:

THEOREM 3. *If P is a prime ideal that has not exclusively zeros of the form $\{0, \dots, 0; \eta_0, \dots, \eta_m\}$ or $\{\xi_1, \dots, \xi_n; 0, \dots, 0\}$, $\overline{P}$ is again prime, and each general zero $\{\xi_1, \dots, \xi_n : \eta_1, \dots, \eta_m\}$ of P yields a general zero $\{\lambda, \lambda\xi_1, \dots, \lambda\xi_n; \mu, \mu\eta_1, \dots, \mu\eta_m\}$ of P where λ and μ are variables* 1)*.*

PROOF. We have already seen (§ 6) that P is prime; in the same way that $\{\lambda, \lambda\xi_1, \dots, \lambda\xi_n; \mu, \mu\eta_1, \dots, \mu\eta_n\}$ is a zero of P (§ 10). If f is a polynomial in R and if $f(\lambda, \lambda\xi_1, \dots, \lambda\xi_n; \mu, \mu\eta_1, \dots, \mu\eta_m) = 0$, we can in the first place decompose f into homogeneous component parts:

$$f = \Sigma f_i \,.$$

If γ_i, δ_i are the degrees of f_i, we have

$$\Sigma f_i\,(\lambda, \lambda\xi_1, \dots \lambda\xi_n; \mu, \mu\eta_1, \dots, \mu\eta_m) = 0$$
$$\Sigma \lambda^{\gamma_i} \mu^{\delta_i} f_i\,(1, \xi_1, \dots, \xi_n; 1, \eta_1, \dots, \eta_m) = 0$$

identically in λ, μ, hence

$$f_i\,(1, \xi_1, \dots, \xi_n; 1, \eta_1, \dots, \eta_m) = 0$$
$$\overline{f_i} = f_i\,(1, x_1, \dots, x_n; 1, y_1, \dots, y_m) \equiv 0\,(\overline{P})$$
$$f_i \equiv 0\,(P_0) \equiv 0\,(P)$$
$$f = \Sigma f_i \equiv 0\,(P).$$

Consequently $\{\lambda, \lambda\xi_1, \dots, \lambda\xi_n; \mu, \mu\eta_1, \dots, \mu\eta_m\}$ is a general zero of P.

§ 12. If r is the dimension of $\overline{P}$, $r+2$ is the dimension of P because by the adjunction of the variables λ, μ the degree of transcendence of the general zero is augmented by 2. For the determination of the number of dimensions of the manifold of P in projective space, the parameters λ, μ are usually considered as not essential and by the number of dimensions not $r+2$, is understood, but r. Accordingly the dimension of the manifold in the twofold-projective space differs 2 unities from the dimension of the corresponding H-ideal 2).

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1) For the notion of a "general zero" of a prime ideal see "Nullstellentheorie der Polynomideale" § 3, Math. Ann. **96** (1926) p. 183.

2) In the ordinary (single-) projective space this difference is 1.

The number r, which for H-ideals generally plays a larger part than the real dimension $r+2$, may be called the *reduced dimension* of P.

Projectively irrelevant ideals P have no reduced dimension, because, in this case $\overline{P}=(1)$, hence P has no dimension. For complete inductions it is sometimes desirable to assign the reduced dimension -1 to the projectively irrelevant ideals (just as we can assign the dimension -1 to the unit ideal and to the empty point class).

The reduced dimension of a primary ideal is defined as that of the corresponding prime ideal, and that of an arbitrary ideal as the largest of the reduced dimensions of the primary components.

II. *The characteristic function of* HILBERT.

§ 13. HILBERT and LASKER have investigated as a function of ϱ the number of independent linear equations which a form of degree ϱ in $x_1, \ldots, x_n$ must satisfy in order to belong to a given H-ideal. The principal result is that for sufficiently large values of ϱ this number can be represented by a whole rational function, the "characteristic function". LASKER gives a few indications how the theorems proved by him can be extended to multiple homogeneous ideals. As these indications are not quite clear in every respect, I shall treat the extension in question again. The methods for the proofs are LASKER's. In order to understand this §, however, it is not necessary to be acquainted with LASKER's investigations.

§ 14. *Definitions.* Let Γ be again a field and $R=\Gamma[x_0, \ldots, x_n; y_0, \ldots, y_m]$. By $\varphi(\varrho, \sigma)$ we indicate the number of power products of degree ϱ in the x, σ in the y, by $\varphi(\varrho, \sigma;\, A)$ the number of linearly independent forms of the degrees ϱ, σ in a given H-ideal A, and finally by $\chi(\varrho, \varrho, A)$ the number of mod. A linearly independent forms of the degrees ϱ, σ or, what amounts to the same, the number of independent linear equations which forms of the degrees ϱ, σ must satisfy in order to belong to the ideal A.

§ 15. Apparently

$$\varphi(\varrho, \sigma)=\varphi(\varrho, \sigma;(1)) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\chi(\varrho, \sigma;\, A)=\varphi(\varrho, \sigma)-\varphi(\varrho, \sigma;\, A) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

§ 16. THEOREM 4. *If A, B are two H-ideals, (A, B) their sum or G. C. M., $[A, B]$ their intersection, we have*

$$\varphi(\varrho, \sigma;(A, B))=\varphi(\varrho, \sigma: A)+\varphi(\varrho, \sigma;\, B)-\varphi(\varrho, \sigma;\, [A, B]) \quad . \quad (3)$$

$$\chi(\varrho, \sigma;(A, B))=\chi(\varrho, \sigma;\, A)+\chi(\varrho, \sigma;\, B)-\chi(\varrho, \sigma: [A, B]) \quad . \quad (4)$$

(3) is at once evident, for the forms of the degrees ϱ, σ in the ideal (A, B) form a linear set which is additively composed of the linear sets

of the forms of the same degrees in A and B; these sets have all forms of the same degrees in $[A, B]$ in common.

(4) follows from (2) and (3).

§ 17. THEOREM 5. *If f is a form of the degrees γ, δ and if f is relatively prime with respect to A (i.e. that $fg \equiv O(A)$ implies $g \equiv O(A)$), we have*

$$\chi(\varrho, \sigma; (A, f)) = \chi(\varrho, \sigma; A) - \chi(\varrho - \gamma, \sigma - \delta; A) \quad . \quad . \quad . \quad (5)$$

PROOF. The form f generates an ideal F. We shall first determine the functions $\varphi(\varrho, \sigma; F)$ and $\varphi(\varrho, \sigma; [A, F])$.

All forms of the degree ϱ, σ in F have the form $f \cdot g$, where g is a form of the degrees $\varrho - \gamma, \sigma - \delta$. Hence

$$\varphi(\varrho, \sigma; F) = \varphi(\varrho - \gamma, \sigma - \delta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The forms of the degree ϱ, σ in $[A, F]$ must lie in F (and consequently have the form $f \cdot g$) as well as in A; this gives

$$fg \equiv 0(A)$$

This congruence implies $g \equiv 0(A)$ and inversely. Hence the number of linearly independent forms $f \cdot g$ of the degrees ϱ, σ in $[A, F]$ is equal to the number of linearly independent forms g of the degrees $\varrho - \gamma, \sigma - \delta$ in A. Hence:

$$\varphi(\varrho, \sigma; [A, F]) = \varphi(\varrho - \gamma, \sigma - \delta; A) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

According to (4) we have:

$$\chi(\varrho, \sigma; (A, F)) = \chi(\varrho, \sigma; A) + \chi(\varrho, \sigma; F) - \chi(\varrho, \sigma; [A, F])$$

according to (2):

$$= \chi(\varrho, \sigma; A) - \varphi(\varrho, \sigma; F) + \varphi(\varrho, \sigma; [A, F])$$

according to (6), (7):

$$= \chi(\varrho, \sigma; A) - \varphi(\varrho - \gamma, \sigma - \delta) + \varphi(\varrho - \gamma, \sigma - \delta; A)$$

according to (2):

$$= \chi(\varrho, \sigma; A) - \chi(\varrho - \gamma, \sigma - \delta; A).$$

§ 18. THEOREM 6. *For projectively irrelevent ideals (i.e. for those ideals A that have no other zeros than $\{0, \ldots, 0 : \eta_0, \ldots, \eta_n\}$ and $\{\xi_1, \ldots, \xi_n ; 0, \ldots, 0\}$ and for sufficiently large ϱ, σ we have:*

$$\chi(\varrho, \sigma; A) = 0.$$

PROOF. All polynomials of the ideal

$$(x_0, \ldots, x_n) \cdot (y_0, \ldots, y_n)$$

become zero in the zeros of A; hence according to HILBERT's theorem of zeros we have for a certain λ

$$(x_0, \ldots, x_n)^\lambda \cdot (y_0, \ldots, y_n)^\lambda \equiv 0(A)$$

i.e. all power products whose degrees are $\geqq \lambda$, lie in A. Consequently the same holds good for arbitrary forms whose degrees are $\geqq \lambda$.

§ 19. THEOREM 7. *If A is an H-ideal and d its reduced dimension, $\chi(\varrho, \sigma; A)$ is represented for high values of ϱ and σ by an expression of the form:*

$$\chi(\varrho, \sigma; A) = \sum_{i+j \leqq d} a_{ij} \binom{\varrho}{i} \binom{\sigma}{j} \quad (\varrho \geqq \varrho_0, \sigma \geqq \sigma_0) \quad . \quad . \quad . \quad (8)$$

with whole rational coefficients a_{ij}.

This expression is called the *characteristic function* of the ideal A. If the binomial coefficients are written in full, it may also be written as a polynomial in ϱ and σ.

PROOF. If $d = -1$, i. e. if A is projectively irrelevant, then according to theorem 6 we have $\chi(\varrho, \sigma; A) = 0$ for high values of ϱ and σ. If, therefore, we consider 0 as a polynomial of the degree -1 in ϱ and σ, the theorem holds good for $d = -1$. In the proof which follows, it will become clear that in fact we can begin the complete induction at $d = -1$.

We shall now assume that $d \geqq 0$ and that the theorem has been proved for all reduced dimensions $< d$. The given ideal A is the intersection of primary ideals

$$A = [Q_1, \ldots, Q_r]$$

and all Q_i have the reduced dimension d at most. If we suppose for a moment that the theorem has already been proved for primary ideals of the same dimension and (if $r > 1$) for ideals with fewer than r primary components, there follows from (4):

$$\chi(\varrho, \sigma; A) = \chi(\varrho, \sigma; [[Q_1, \ldots, Q_{r-1}], Q_r]])$$
$$= \chi(\sigma, \sigma; [Q_1, \ldots, Q_{r-1}]) + \chi(\varrho, \sigma; Q_r) - \chi(\varrho, \sigma; ([Q_1, \ldots, Q_{r-1}], Q_r))$$

The former two terms on the right-hand side refer to ideals with fewer primary components than A, and the latter term to an ideal of fewer dimensions, hence all functions, on the right-hand side have the form (8), so that the same is valid for the left-hand side. It is, therefore, only necessary to prove the theorem for primary ideals. Let A be primary and let P be the corresponding prime ideal. We choose two linear forms $l_1 = \sum_0^n u_i x_i$, $l_2 = \sum_0^m v_i y_i$ that do not belong to P. The ideals (A, l_1) and (A, l_2) have the reduced dimensions $d-1$ at most, and the forms l_1, l_2 are relatively prime with respect to A. Hence (5) holds good:

$$\left.\begin{aligned} \chi(\varrho, \sigma; (A, l_1)) &= \chi(\varrho, \sigma; A) - \chi(\varrho-1, \sigma; A) \\ \chi(\varrho, \sigma; (A, l_2)) &= \chi(\varrho, \sigma; A) - \chi(\varrho, \sigma-1; A) \end{aligned}\right\} \quad . \quad . \quad . \quad (9)$$

According to the supposition we have for $\varrho \geqq \varrho_0, \sigma \geqq \sigma_0$:

$$\chi(\varrho, \sigma; (A, l_1)) = \sum_{i+j \leqq d-1} b_{ij} \binom{\varrho}{i} \binom{\sigma}{j}$$

$$\chi(\varrho, \sigma; (A, l_2)) = \sum_{i+j \leqq d-1} c_{ij} \binom{\varrho}{i} \binom{\sigma}{j}$$

On summation we get from (9_1):

$$\begin{aligned}\chi(\varrho, \sigma : A) - \chi(\varrho_0, \sigma ; A) &= \sum_{\lambda=\rho_0+1}^{\rho} \chi(\lambda, \sigma ; (A, l_1)) \\ &= \sum_{i+j \leqq d-1} \; \sum_{\lambda=\rho_0+1}^{\rho} b_{ij} \binom{\lambda}{i}\binom{\sigma}{j} \\ &= \sum_{i+j \leqq d-1} b_{ij} \left\{ \binom{\varrho+1}{i+1} - \binom{\varrho_0+1}{i+1} \right\} \binom{\sigma}{j}\end{aligned}$$

and in the same way from (9_2):

$$\chi(\varrho_0, \sigma ; A) - \chi(\varrho_0, \sigma_0 ; A) = \sum_{i+j \leqq d-1} c_{ij} \binom{\varrho_0}{i} \left\{ \binom{\sigma+1}{j+1} - \binom{\sigma_0+1}{j+1} \right\}.$$

If, finally, we add these two equations we find:

$$\left.\begin{aligned}\chi(\varrho, \sigma ; A) = \chi(\varrho_0, \sigma_0 ; A) + \sum_{i+j \leqq d-1} b_{ij} \binom{\varrho+1}{i+1}\binom{\sigma}{j} - \\ - \sum_{i+j \leqq d-1} b_{ij} \binom{\varrho_0+1}{i+1}\binom{\sigma}{j} + \sum_{i+j \leqq d-1} c_{ij} \binom{\varrho_0}{i}\binom{\sigma+1}{j+1} - \sum_{i+j \leqq d-1} c_{ij} \binom{\varrho_0}{i}\binom{\sigma_0+1}{j+1}\end{aligned}\right\} (10)$$

The right hand side has in fact the form (8), if we take into consideration that $\binom{\varrho+1}{i+1} = \binom{\varrho}{i+1} + \binom{\varrho}{i}$, etc.

§ 20. Among the numbers a_{ij} we shall pay special attention to those for which $i + j = d$, i.e. to the coefficients of the terms of highest degree in the characteristic function. These a_{ij} are called the *degrees* of the ideal A; their geometrical signification will appear later. For the degrees we introduce the symbol $a_{ij}(A)$.

From the proof given above the following theorems follow directly:

THEOREM 8. *Any degree* a_{ij} *of an H-ideal is the sum of the corresponding degrees of the primary components with the same reduced dimension.*

THEOREM 9. *If* Q *primary,* P *the corresponding prime ideal,* l_1 *a linear form in* x_i, *and* $l_1 \equiv\!\!|\!\!\equiv 0\,(P)$, *we have*

$$a_{ij}(Q) = a_{i-1,j}(Q, l_1) \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In the same way if l_2 *is a linear form in the* y_i *and* $l_2 \equiv\!\!|\!\!\equiv 0\,(P)$:

$$a_{ij}(Q) = a_{i,j-1}\,(Q, l_2) . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

A generalisation of theorem 9 is:

THEOREM 10. *If* Q *is a primary ideal,* P *the corresponding prime ideal,* f *a form of the degrees* γ, δ, *and* $f \equiv\!\!|\!\!\equiv 0\,(P)$, *we have:*

$$a_{ij}\,(Q, f) = \gamma \,.\, a_{i+1,j}\,(Q) + \delta \,.\, a_{i,j+1}\,(Q)$$

PROOF. The suppositions of theorem 5 are fulfilled, accordingly (5) is valid for $A = Q$. If we expand the two members of (5) according to theorem 7 with respect to binomial coefficients, and if we compare the coefficients of $\varrho^i\,\sigma^j$ on the two sides, we find the result in question directly.

§ 21. THEOREM 11. *The degrees of an H-ideal are* $\geqq 0$.

PROOF. For $d=0$ the characteristic function becomes a constant a_{00}. In this case the ideal has a finite number of zeros in the projective space (with coordinates from Γ or a field containing Γ). There are polynomials of an arbitrarily high degree which are in at least one of these zeros $\neq 0$, and which, therefore, do not belong to the ideal. Hence the characteristic function a_{00} is in this case even >0.

Now suppose the theorem to be valid for arbitrary ideals of less than d reduced dimensions. In this case for primary ideals of d reduced dimensions the theorem follows by formulas (11) and (12). For arbitrary ideals of d dimensions the same follows from theorem 8.

§ 22. THEOREM 12. *The characteristic function of an H-ideal A is the same as that of the ideal* A_0 *that is found by omitting the projectively-irrelevant primary components from A* (*cf.* § 8).

PROOF. Put

$$M=[Q_1, \ldots, Q_r],$$

$$Q_1, \ldots, Q_s \text{ projectively irrelevant,}$$

hence

$$M_0=[Q_{s+1}, \ldots, Q_r].$$

If we put $M_1=[Q_1, \ldots, Q_s]$, we have $M=[M_0, M_1]$.

Now (4) implies:

$$\begin{aligned}\chi(M;\varrho,\sigma)&=\chi(M_0;\varrho,\sigma)+\chi(M_1;\varrho,\sigma)-\chi((M_0;M_1);\varrho,\sigma)\\ &=\chi(M_0;\varrho,\sigma) \qquad (\varrho\geqq\varrho_0,\ \sigma\geqq\sigma_0),\end{aligned}$$

because the characteristic functions of the projectively-irrelevant ideals M_1 and (M_0, M_1) are zero (§ 18).

III. *Series of Composition of Primary Ideals.* [1])

§ 23. Let Q be a primary ideal in a domain (German: „Ring") R, let P be the corresponding prime ideal, and suppose $P \neq R$. By a *series of composition* of Q we understand a finite series of primary ideals

$$(Q_0=R); \quad Q_1=P; \quad Q_2; \ldots; \ Q_l=Q,$$

(where the first term $Q_0=R$ may be reckoned to belong to the series or not) with the following properties:

1) Q_i is primary and P is the corresponding prime ideal $(i=1, \ldots, l)$

2) $Q_{i+1} \neq Q_i \qquad (i=1, \ldots, l-1)$

3) $Q_{i+1} \equiv 0\ (Q_i) \qquad (i=1, \ldots, l-1)$.

4) It is impossible to insert a primary ideal between Q_i and Q_{i+1} so that the properties 1), 2), 3) remain valid.

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[1]) This § is a partial elaboration of ideas of E. NOETHER's; cf. her address in Göttingen on the numbers of HILBERT, Jahresbericht D.M.V. 34, (1925), p. 101. Cf. also W. KRULL, verallgemeinerte Abelsche Gruppen, Sitzungsberichte Heidelberger Akad. 1926.

We shall now prove *the existence of a series of composition for any primary ideal* Q on the supposition that the corresponding prime ideal P has *a finite basis.*

§ 24. We shall first simplify the problem by passing from the ring R to the ring of restclasses $R' = R/Q$. Every ideal A, divisor of Q, gives rise to an ideal $A' = A/Q$ in R'. Especially P leads to a prime ideal P', and any primary ideal Q_i corresponding to P leads to a primary ideal Q'_i corresponding to P'. Q itself leads to $Q' = Q/Q$: the zero ideal in R'. Conversely A is uniquely defined by A': members of A are the members of all restclasses with respect to Q in A'. Accordingly it is sufficient to prove the existence of a series of composition

$$Q'_1 = P', \quad Q'_2, \ldots, Q'_l = (0).$$

The basis of P leads to a basis of P'.

We shall now again pass from the ring R' to a domain R^* consisting of all fractions $\frac{a}{b}$, $b \equiv\mid\equiv 0(P')$. We put $\frac{a}{b} = \frac{a'}{b'}$, if $a\,b' = a'\,b$.

Addition and multiplication in this domain are as usually defined by the formulas

$$\frac{a}{b} + \frac{a'}{b'} = \frac{a\,b' + b\,a'}{b\,b'}$$

$$\frac{a}{b} \cdot \frac{a'}{b'} = \frac{a\,a'}{b\,b'}$$ [1]

The members $b \equiv\mid\equiv 0\,(P)$ are no zero factors, for as the zero ideal Q' is primary and P the corresponding prime ideal, the relations $bc = 0$, $c \neq 0$ always implie $b \equiv 0\,(P)$.

The prime ideal P' generates a prime ideal P^* in R^* consisting of all fractions $\frac{a}{b}$; $a \equiv 0\,(P')$. In the same way any primary ideal Q'_i in R' of which the corresponding prime ideal is P', generates a primary ideal Q^*_i in R^* of which the corresponding prime ideal is P^*. All this can be verified without difficulty. Conversely Q'_i is defined unambiguously by Q^*_i : Q'_i consists of all numerators of the fractions that appear in Q^*_i [2]. It is, therefore, sufficient to prove the existence of a series of composition

$$Q^*_1 = P^*; \quad Q^*_2; \ldots; \quad Q^*_l = Q^*$$

The ring R^* has this advantage over R that it has the following properties, which R did not necessarily possess:

I. There is a unity, viz. $\frac{b}{b}$ (b arbitrary, $\equiv\mid\equiv 0\,(P')$).

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[1] E. STEINITZ, Algebr. Theorie der Körper § 3, J. f. M. **137**, (1910). H. GRELL, Beziehungen zwischen den Idealen verschiedener Ringe § 1, (5), Math. Ann. **97**, (1927), p. 490.

[2] H. GRELL, loc. cit., § 6.

II. The equation $a \,.\, x = 1$ is always solvable for $a \not\equiv 0\,(P^*)$.

III. Q^* is the zero ideal.

After multiplication by $\frac{b}{b}$ the basis elements of P' can be used as basis elements for P^*.

We shall henceforward omit the asterisks and suppose R to possess the properties I, II, III.

Now let ϱ be the exponent of the zero ideal Q, i.e. the smallest number with the property $P^\varrho = 0$.

If we prove that a series of composition from P^λ to $P^{\lambda+1}$ $(\lambda = 1, \dots, \varrho - 1)$ is possible, we need only place all these series one after another to obtain a series of composition from P to Q.

The powers of P, $P^2, \dots . P^\varrho$ are all different, for $P^\lambda = P^{\lambda+1}$ would implie $P^{\varrho-1} = P^\varrho = 0$.

Further P^λ has a finite basis $(f_1, \dots, f_r)$, so that we can write:

$$P^\lambda = (P^{\lambda+1}, f_1, \dots, f_r) \quad . \; . \; . \; . \; . \; . \; . \; . \tag{1}$$

We shall suppose as many f_i's as possible to be omitted on the right-hand side, so that if we omitted one more of them (1) would become fals. Now I assert that the series of ideals

$$Q_0^{(\lambda)} = (P^{\lambda+1}, f_1, \dots, f_r) = P^\lambda$$

$$Q_1^{(\lambda)} = (P^{\lambda+1}, f_1, \dots, f_{r-1})$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$Q_{r-1}^{(\lambda)} = (P^{\lambda+1}, f_1)$$

$$Q_r^{(\lambda)} = P^{\lambda+1}$$

is a series of composition from P^λ to $P^{\lambda+1}$. We must, therefore, prove the properties 1—4 (§ 23).

1. It appears thus that $Q_i^{(\lambda)}$ is primary and that P is the corresponding prime ideal:

a. From $ab \equiv 0\,(Q_i^{(\lambda)})$ and $b \neq 0\,(P)$ we find on multiplication by b^{-1}:

$$a \equiv 0\,(Q_i^{(\lambda)})$$

b. $a \equiv 0\,(P)$ implies $a^\varrho \equiv 0\,(Q) \equiv 0\,(Q_i^{(\lambda)})$.

c. $a^\sigma \equiv 0\,(Q_i^{(\lambda)})$ implies $a^\sigma \equiv 0\,(P)$, hence $a \equiv 0\,(P)$.

2. $Q_i^{(\lambda)} \neq Q_{i+1}^{(\lambda)}$ is evident, for else

$$(Q^{(\lambda)}, f_1, \dots, f_i) = (Q^{(\lambda)}, f_1, \dots, f_{i-1})$$

and then f_i might be omitted from the equation (1).

3. $Q_{i+1}^{(\lambda)} \equiv 0\,(Q_i^{(\lambda)})$ is evident.

4. If it were possible to insert an ideal Q' between $Q_i^{(\lambda)}$ and $Q_{i+1}^{(\lambda)}$:

$$Q_{i+1}^{(\lambda)} \equiv 0\,(Q') \equiv 0\,(Q_i^{(\lambda)})$$

$$Q_{i+1}^{(\lambda)} \neq Q' \neq Q_i^{(\lambda)}$$

Q' would contain a member f' which does not lie in $Q^{(\lambda)}_{i+1}$. This member lies in $Q^{(\lambda)}_i = (Q^{(\lambda)}_{i+1}, f_{r-i})$, hence

$$f' \equiv h \,.\, f_{r-i} \quad (Q^{(\lambda)}_{i+1}) \quad \dots\dots\dots\dots \quad (2)$$

If h lies in P we have:

$$h \,.\, f_{r-i} \equiv 0\,(P P^{\lambda}) \equiv 0\,(P^{\lambda+1}) \equiv 0\,(Q^{(\lambda)}_{i+1})$$

and, therefore, $f' \equiv 0\,(Q^{(\lambda)}_{i+1})$, against the supposition.

If h does not lie in P we can multiply (2) by h^{-1}:

$$f_{r-i} \equiv h^{-1} f' \quad (Q^{(\lambda)}_{i+1})$$

$$f_{r-i} \equiv 0 \qquad (Q')$$

$$Q^{(\lambda)}_i = (Q^{(\lambda)}_{i+1}, f_{r-i}) \equiv 0\,(Q')$$

$$Q^{(\lambda)}_i = Q',$$

against the supposition[1]).

§ 25. According to the proof the series of composition constructed by us has the following properties besides the postulated properties 1—4:

5. It is even impossible to insert an arbitrary ideal Q' with the properties $Q_{i+1} \equiv 0\,(Q') \equiv 0\,(Q_i)$, $Q_{i+1} \neq Q' \neq Q_i$ between Q_i and Q_{i+1}, whether Q' is primary or not.

6. $$P Q_i \equiv 0\,(Q_{i+1})$$

7. $$Q_i = (Q_{i+1}, f_i)$$

One might think that these properties are only consequences of the construction we have followed and that perhaps another series of composition does not possess these properties. We shall prove that, on the contrary, *these 3 properties are valid for any series of composition*, where we need not make the suppositions I, II, III for our ring R, but only the much less far reaching suppositions:

I*. *R contains a unity;*

II*. *The prime ideal P does not contain any true divisors besides R.*

PROOF of 5. Let Q' be an arbitrary ideal, $Q_{i+1} \equiv 0\,(Q') \equiv 0\,(Q_i)$. In order to reduce everything to 4. we shall prove that Q' most necessarily be primary and that P is the corresponding prime ideal.

a. The relations $ab \equiv 0\,(Q')$ and $b \not\equiv 0\,(P)$ imply in the first place $(b, P) \neq P$, hence $(b, P) = R$; accordingly any member of R has the form $h\,b + p\;(p \equiv 0\,(P))$; especially:

$$1 = h\,b + p$$

$$a = h\,a\,b + a\,p \equiv a\,p \quad (Q')$$

$$a \equiv a\,p \equiv a\,p^2 \equiv, \dots \equiv a\,p^{\rho} \equiv 0\,(Q').$$

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[1]) In the original Dutch paper the proofs of this § were not correct, at least not generally valid. A correction will be added. The English text given above is correct.

b. $a \equiv 0\,(P)$ implies $a^{\rho} \equiv 0\,(Q) \equiv 0\,(Q')$.
c. $a^{\sigma} \equiv 0\,(Q')$ implies $a^{\sigma} \equiv 0\,(P)$, hence $a \equiv 0\,(P)$.

PROOF of 6. We have

$$Q_{i+1} \equiv 0\,(PQ_i\,,\; Q_{i+1}) \equiv 0\,(Q_i)$$

hence, according to 5:

$$\begin{array}{ll} \text{either} & (PQ_i\,,\; Q_{i+1}) = Q_i \\ \text{or} & (PQ_i\,,\; Q_{i+1}) = Q_{i+1} \end{array} \Bigg\} \quad \ldots\ldots\ldots \quad (3)$$

In the first case

$$PQ_i \equiv Q_i\;(Q_{i+1})$$
$$Q_i \equiv PQ_i \equiv P^2Q_i \equiv \ldots \equiv P^{\rho}\,Q_i \equiv 0\,(Q_{i+1})$$
$$Q_i = Q_{i+1}$$

which is impossible. Hence only the second alternative of (3) remains; it implies

$$PQ_i \equiv 0\,(Q_{i+1}), \quad \text{q. e. d.}$$

PROOF of 7. Let f_i be a member of Q_i that does not belong to Q_{i+1}; then

$$(Q_{i+1}, f_i) \neq Q_{i+1}$$

and consequently, according to 5.:

$$(Q_{i+1}, f_i) = Q_i, \qquad \text{q.e.d.}$$

§ 26. According to the well known theorem of JORDAN—HÖLDER the existence of one single series of composition implies *that all series of composition with the properties* 2, 3, 5 *have the same lengths* l. In a domain with the properties I*, II* any series of composition for primary ideal has the properties 2, 3, 5 and all have, therefore, the same length; this result may be directly extended to arbitrary rings by means of the transformation $R \rightarrow R^*$.

The theorem of JORDAN—HÖLDER, however, is not strictly necessary for what follows; we avoid the use of this theorem by the following definition:

The *length* of a primary ideal Q is the length of the shortest series of composition for this ideal. In reality l is at the same time the length of any series of composition.

§ 27. *If a series of composition has the properties* 1—7, *the following modulus-isomorphism is valid*:

$$Q_i/Q_{i+1} \cong R/P \qquad (i = 0, \ldots, l-1) \quad \ldots\ldots \quad (4)$$

where both members must be considered as R-moduli 1) 2)). In terms of

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1) An R-module is a set of members which together with a and b also contain $a+b$ and $a-b$ and together with a always $g\,.\,a$ where g is an arbitrary member of R. In our case those members are restclasses mod. Q_{i+1} resp. mod. P. A module-isomorphism is a one-one correspondence $a \rightarrow \bar{a}$ with the properties

$$a + b \rightarrow \bar{a} + \bar{b}, \quad g\,.\,a \rightarrow g\,.\,\bar{a}.$$

2) Cf. H. GRELL, Ordnungen in Zahl- und Funktionenkörpern, Math. Ann. **97**, (1927) p. 540.

the theory of groups we have to do with series of composition where all factor groups are isomorphic.

PROOF. As $Q_i = (Q_{i+1}, f_i)$ all members of Q_i are mod. Q_{i+1} multiples of f_i, i.e. any member of Q_i is $\equiv h \, . \, f_i \; (Q_{i+1})$. To any member h of R there corresponds a member $h \, . \, f_i$ of Q_i; if $h_1 - h_2 \equiv 0 \, (P)$, we have $h_1 f_i - h_2 f_i \equiv 0 \, (P \, Q_i) \equiv 0 \, (Q_{i+1})$. If conversely $h_1 f_i - h_2 f_i \equiv 0 \, (Q_{i+1})$ we have, according to $f_i \equiv\!\!|\!\!\equiv 0 \, (Q_{i+1})$,

$$h_1 - h_2 \equiv 0 \, (P).$$

If, therefore, for the members h of R we choose the congruence mod. P as definition of equality and for the members $h \, . \, f_i$ of Q_i the congruence mod. Q_{i+1}, the correspondence $h \to h f_i$ is a one-one correspondence. To $h_1 + h_2$ there corresponds $h_1 f_i + h_2 f_i$ and to $g \, . \, h$ there corresponds $g \, . \, h f_i$. In this way the isomorphism (4) for $i = 1, \ldots, l-1$ is proved. For $i = 0$ the members of (4) are identical.

IV. *The Degrees of an Ideal.*

§ 28. Let Q be a zero-dimensional primary ideal in $R = \Gamma[x_1, \ldots, x_n; y_1, \ldots, y_m]$. The H-ideal Q_0 equivalent to Q (§ 5) in $\Gamma \, [x_0, \ldots, x_n \; y_0, \ldots, y_m]$ has the reduced dimension zero (§ 13), and its characteristic function is, therefore, a constant $a_{00} \, (Q_0)$ that is at the same time the degree of Q_0. By the substitution $x_0 = 1$, $y_0 = 1$ the forms of the degrees ϱ, σ in Q_0 are transformed into all polynomials of $\bar{Q}$ of which the degrees are $\leqq \varrho, \leqq \sigma$; hence the number of the linearly independent polynomials of the degrees $\leqq \varrho, \leqq \sigma$ in $\bar{Q}$ is equal to that of the polynomials of the degrees ϱ, σ in Q. In the same way the number of mod. Q_0 linearly independent forms of the degrees ϱ, σ is equal to the number of mod. Q linearly independent polynomials of the degrees $\leqq \varrho, \leqq \sigma$ in R. As soon as ϱ, σ exceed a certain limit, the former number is constant viz. equal to the characteristic function $a_{00} \, (Q_0)$; consequently the same thing holds for the latter number:

There are only a finite number of mod. Q linearly independent polynomials in R and this number is equal to $a_{00} \, (Q_0)$.

We can also express this in the following way:

The domain of restclasses R/Q is a modulus of finite rank[1]) *relative to Γ.*

§ 29. The same result can also be reached in a different way without making use of H-ideals and characteristic functions. At the same time we shall find a remarkable relation between lengths and degrees.

Let in the first place P be a zero-dimensional prime ideal and let $\{\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_m\}$ be its general zero. In this case all ξ_1, η are algebraic relative to Γ, hence the field $\Gamma(\xi, \eta)$ has a finite degree relative te Γ. Let this degree be γ. In consequence of the isomorphism

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[1]) Rank = Maximum number of elements linearly independent with respect to Γ.

$$\Gamma(\xi, \eta) \cong R/P\,^{1})$$

also R/P has a finite degree relative to Γ.

Now let Q be a primary ideal corresponding to P. According to § 24 there exists a series of composition:

$$(Q_0 = R);\; Q_1 = P;\; Q_2; \ldots ;\; Q_l = Q.$$

As P (being a zero-dimensional prime ideal) has no true divisors the series of composition has the properties 1—7 of § 25, and according to § 26 these implie

$$Q_i/Q_{i+1} \cong R/P$$

Hence also:

$$\text{Rank } Q_i/Q_{i+1} = \text{Rank } R/P$$

Summation from $i = 0$ to $i = l - 1$ gives

$$\text{Rank } Q_0/Q_l = l.\ \text{Rank } R/P,$$
$$\text{Rank } R/Q = l.\ \text{Rank } R/P,$$

by which it is proved that the rank of R/Q is finite. The rank of R/Q will be called the *degree* of Q. At the same time we have found the following relation:

The degree of a zero-dimensional primary ideal is equal to this length multiplied by the degree of the corresponding prime ideal. [2])

§ 30. In order to extend these theorems to more-dimensional ideals, we shall first prove a few auxiliary theorems.

LEMMA. *Let* Q *be a primary ideal in* $\Gamma[x_1, \ldots, x_n;\, y_1, \ldots, y_n]$, P *the the corresponding prime ideal* $\{\xi_1, \ldots, \xi_n, \ldots\}$ *a general zero of* P. *If now* ξ_1 *is transcendent (variable) and* t *a variable which we adjoin to* Γ, *then the ideal* $(Q, x_1 - t)$ *in* $\Gamma(t)[x, y]$ *is also primary, and* $(P, x_1 - t)$ *is the corresponding prime ideal. The length of* $(Q, x_1 - t)$ *is equal to that of* Q.

PROOF. In the first place it is easily seen that a polynomial $f(x, y)$ which is independent of t, can only be $\equiv 0\,(Q, x_1 - t)$, if it is $\equiv 0\,(Q)$. To see this we need only write the congruence $f \equiv 0\,(Q, x_1 - t)$ in full as an equation between polynomials x, y and t, make the fractions disappear by multiplying both members by the denominator $n(t)$, put $t = x_1$ and divide the congruence $n(x_1) . f(x, y) \equiv 0\,(Q)$ arising in this way by $n(x_1)$, which is allowed, because we have $n(\xi_1) =\mid= 0$ and hence $n(x_1) \equiv\mid\equiv 0\,(P)$.

In order to prove that $(Q, x_1 - t)$ is primary and that $(P, x_1 - t)$ is the corresponding prime ideal, we must show that:

1. $ab \equiv 0\,(Q, x_1 - t)$ and $a \equiv\mid\equiv 0\,(Q, x_1 - t)$ imply $b \equiv 0\,(P, x_1 - t)$.
2. $b \equiv 0\,(P, x_1 - t)$ implies $b^{\rho} \equiv 0\,(Q, x_1 - t)$, and conversely.

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[1]) Cf. „Nullstellentheorie" loc. cit. § 3, 2.

[2]) Cf. H. GRELL, loc. cit. § 3, Satz 2.

We can suppose the polynomials a, b that are rational in t, to be whole and rational in t. Further we may everywhere replace t mod. (x_1-t) by x_1. If this is done, we have only polynomials in the x and y and all our congruences $ab\equiv(Q, x_1-t)$, etc. may, therefore, also be read as: $ab\equiv 0\,(Q)$, etc. Thus 1. and 2. only express that Q is primary and that P is the corresponding prime ideal.

In order to prove that the length of (Q, x_1-t) is the same as that of Q, we must show that there exists a one-one correspondence between the primary ideals Q' in $\Gamma[x, y]$ for which

$$Q\equiv 0\,(Q')\equiv 0\,(P) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the primary ideals Q'' in $\Gamma(t)\,[x, y]$, for which

$$(Q, x_1-t)\,Q\equiv 0\,(Q'')\equiv 0\,(P, x_1-t). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and that this one-one correspondence leaves intact the relation divisor-multiple.

If Q' is given, we put $Q''=(Q', x_1-t)$. We know already that in this case Q'' is primary and that (P, x_1-t) is the corresponding prime ideal. It is evident that (2) holds good. From $f(x, y)\equiv 0\,(Q', x_1-t)$ there follows $f(x, y)\equiv 0\,(Q')$ (see above), hence Q' consists of all polynomials in Q'', that are independent of t; accordingly Q' is defined uniquely by Q''. It is evident that $Q'_1\equiv 0\,(Q'_2)$ implies $Q''_1\equiv 0\,(Q''_2)$, and conversely. We have still to prove that for Q'' we may choose any given primary ideal that satisfies (2).

Let Q'' be a primary ideal which satisfies (2) and of which, consequently, the corresponding prime ideal is (P, x_1-t). Let Q' be the aggregate of all polynomials in Q'' independent of t. Any polynomial of Q'' after being made whole in t by multiplication by the denominator $n(t)$, may be replaced mod. (x_1-t) by a polynomial that no longer depends on t, accordingly by a polynomial of Q'. Hence $Q''=(Q', x_1-t)$. It is seen easily that Q' is primary and P the corresponding prime ideal.

§ 31. This lemma may be transformed into a theorem on H-ideals by means of the methods of Part I:

Let Q be a primary H-ideal in $\Gamma[x_0, \ldots, x_n; y_0, \ldots, y_n]$, P the corresponding prime ideal, $\{\lambda, \lambda\xi_1, \ldots\}$ a general zero of P. Let ξ_1 be transcendent relative to Γ. If we adjoin a variable t to Γ and if we put

$$Q^{(1)}=(Q, x_1-tx_0),$$

then *$Q_0^{(1)}=$ the ideal of the polynomials f for which*

$$x_0^{\varrho}\,y_0^{\sigma}\,f\equiv 0\,(Q^{(1)}),$$

then $Q_0^{(1)}$ is primary and the ideal $P_0^{(1)}$ defined in an analogous way is the corresponding prime ideal. The length of $Q_0^{(1)}$ is equal to that of Q.

PROOF. Let us construct the non-homogeneous ideals $\overline{P}$, $\overline{Q}$, etc. as indicated in § 4. We have (§ 6):

$$\overline{Q^{(1)}}=(\overline{Q}, x_1-t)$$

and the H-ideal equivalent to $\overline{Q^{(1)}}$ is $Q_0^{(1)}$. The same holds good for $\overline{P^{(1)}}$ and $P_0^{(1)}$. As the transformation to equivalent H-ideals leaves intact the properties prime, primary, corresponding prime and divisibility, we reduce all the properties that are to be proved for $Q_0^{(1)}$, $P_0^{(1)}$ to the same properties for $\overline{Q^{(1)}}$, $\overline{P^{(1)}}$, so that everything is reduced to former lemma.

We are now able to generalize the theorems of § 29 to more dimensions:

§ 32. *The degrees a_{ij} of a primary H-ideal Q are equal to the length of Q multiplied by the corresponding degrees of the corresponding prime ideal.*

Proof. First of all let the reduced dimension of Q be equal to zero, hence the characteristic function a constant a_{00}, and at the same time the degree. If we then pass to inhomogeneous ideals $\overline{Q}$, $\overline{P}$, according to § 28 the degree of Q is equal to that of $\overline{Q}$ (i.e. to the rank of $\overline{R/Q}$), and the degree of P is equal to that of $\overline{P}$; further the correspondence between the primary ideals Q corresponding to P and the primary ideals $\overline{Q}$ corresponding to $\overline{P}$ is a one-one correspondence; consequently the length of Q is equal to that of $\overline{Q}$. If l represents this length, according to § 29 we have:

$$\text{Degree } \overline{Q} = l \,.\, \text{Degree } \overline{P}$$

and, therefore:

$$\text{Degree } Q = l \,.\, \text{Degree } P.$$

We shall now suppose the theorem to be proved for all reduced dimensions $< d$ and the reduced dimension of Q to be equal to d $(d > 0)$. Let $\{\lambda, \lambda\xi_1, \ldots, \lambda\xi_n; \mu, \mu\eta_1, \ldots\}$ be a general zero of P. As $d > 0$, one of the ξ_i or η_k must be transcendent; suppose e.g. ξ_1 transcendent. On account of § 20, if t is a variable, we have:

$$a_{ij}(Q) = a_{i-1,j}(Q, x_1 - tx_0) = a_{i-1,j}(Q^{(1)})$$
$$a_{ij}(P) = a_{i-1,i}(P, x_1 - tx_0) = a_{i-1,j}(P^{(1)})$$

According to § 22 the characteristic functions of $Q^{(1)}$, $P^{(1)}$ are the same as those of $Q_0^{(1)}$, $P_0^{(1)}$. Hence:

$$a_{ij}(Q) = a_{i-1,j}(Q_0^{(1)}); \quad a_{ij}(P) = a_{i-1,j}(P_0^{(1)})$$

According to § 31 the length of $Q_0^{(1)}$ is equal to that of Q, hence $= l$. According to the supposition of induction we have for the numbers on the right hand sides the relation

$$a_{i-1,j}(Q_0^{(1)}) = l \,.\, a_{i-1,j}(P_0^{(1)}),$$

which, accordingly, must also be valid for the numbers on the left hand-sides. Thus the theorem is proved for all $a_{ij}(Q)$ with $i > 0$. There remains $a_{0d}(Q)$.

By interchanging x and y a_{0d} can be treated in exactly the same way as formerly a_{d0}, provided one of the η is transcendent. If this is not the case only a finite number of proportions $1 : \eta_1 ; \ldots : \eta_n$ come into consideration for the zeros of Q. If we now choose a linear form $l(y)$ that does not become zero for any of these values, the ideals $(Q, l(y))$ and likewise $(P, l(y))$ become projectively irrelevant, hence

$$a_{0j}(Q) = a_{0,j-1}(Q, l(y)) = 0$$
$$a_{0j}(P) = a_{0,j-1}(P, l(y)) = 0.$$

In this case the assertion reduces to the triviality

$$0 = l \, . \, 0.$$

Thus the theorem is generally proved.

V. *The Geometrical Signification.*

§ 33. We have seen that the degrees of an H-ideal are equal to the sums of the corresponding degrees of the primary components of the highest dimension (§ 20) and that the degrees of these primary components are equal to their lengths multiplied by the degrees of the corresponding prime ideals (§ 32). The degrees of these prime ideals have a simple geometrical signification:

If M is the manifold of P in the projective $P_{n,m}$, then $a_{ij}(P)$ is the number of points of intersection of this manifold with a linear space that is given by i general linear equations in the x and j general linear equations in the y.

The coefficients of these equations must be considered as independent variables $u_1, u_2, \dots$ and the coordinates of the points of intersection as algebraic functions of these variables, hence as members of a suitably chosen field Ω containing $\Gamma(u_1, u_2, \dots)$. We suppose that we have to do with the normal case that Ω is a field of the first kind [1]).

PROOF. Let P be first an ideal of the reduced dimension zero; in this case $i = j = 0$, hence it remains to prove that the degree of P is equal to the number of zeros in a suitable chosen field containing Γ. If we number the coordinates in such a way that $x_0 \equiv\!|\!\equiv 0\ (P)$, $y_0 \equiv\!|\!\equiv (P)$, we can introduce non-homogeneous coordinates for all zeros and instead of P consider the corresponding non-homogeneous (zero-dimensional) ideal $\overline{P}$ (§ 4). The degree of $\overline{P}$ is the same as that of P and is equal to the rank of $\overline{R}/\overline{P}$ (§ 28) and accordingly equal to the degree of the field $\Gamma(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_m)$. Zeros of $\overline{P}$ are the systems $\{\xi_1^{(\alpha)}, \dots, \eta_n^{(\alpha)}\}$ conjugated with $\{\xi_1, \dots, \xi_n, \dots, \eta_m\}$ in a GALOISian extension Ω of Γ. According to a well known theorem in the theory of GALOIS the number of these conjugated systems is equal to the degree of the field $\Gamma(\xi_1, \dots, \eta_m)$, hence equal to the degree of P, q. e. d.

Let us suppose this theorem to be proved for all ideals of reduced dimension $< d$, and P to have the reduced dimension d. We shall first intersect the manifold M with one single linear form

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[1]) I.e. that an equation irreduciable in $\Gamma(u_1, u_2, \dots)$ has no double roots in Ω.

$$l = \sum_0^n u_k x_k \,.$$

Let $\{\lambda, \lambda \xi_1, \ldots ; \ldots \mu \eta_m\}$ be a general zero of P. Let at least one ξ_i be transcendent (otherwise (P, l) becomes projectively irrelevant as in the latter case of § 32, and our theorem becomes trivial). Then also $\Sigma\, u_k \xi_k$ is transcendent relative to $\Gamma(u_1, \ldots, u_n)$. We shall now adjoin $u_1, \ldots, u_n$ to Γ and we shall introduce $\sum_1^n u_k x_k = x_1^*$ as a new coordinate instead of x_1; in this case the general zero of P becomes

$$\{\lambda, \lambda\xi_1^*, \ldots, \lambda\xi_n \;\; ; \;\; \mu, \ldots, \mu\,\eta_m\} \quad ; \quad \xi_1^* = \sum_1^n u_k\, \xi_k$$

and we have

$$l = u_0\, x_0 + x_1^* = x_1^* - t\, x_0 \qquad (t = -\, u_0).$$

All the conditions of the auxiliary theorem of § 31 are now fulfilled. If, therefore, we put

$$P^{(1)} = (P, l) = P, (P_1^* - t\, x_0)$$

$P_0^{(1)}$ is prime. In the same way as in § 32 we find

$$a_{ij}\,(P) = a_{i-1,j}\,(P_0^{(1)}).$$

According to the supposition of induction $a_{i-1,j}\,(P_0^{(1)})$ is the number of points of intersection of the manifold of $P_0^{(1)}$ with a linear space, given by $i-1$ general linear forms in the x, and j linear forms in the y. But the manifold of $P_0^{(1)}$ arises itself from that of P by intersection with the general linear form l. Thus the theorem is proved.

§ 34. Let again M be an algebraic manifold of r dimensions defined by a prime ideal P. By the *degrees* $a_{ij}(M)$ $(i + j = r)$ we understand the degrees $a_{ij}(P)$, i.e. the numbers of points of intersection of M with certain linear spaces (§ 23). We now put the question: *what can be said of the degrees of the intersection of M with a spread $f(x, y) = 0$?*

What the answer about must be may be found by considering a surface S in the ordinary projective space P_3. The intersection of S with another surface $f(x) = 0$ decomposes into different irreducible components whose degrees, multiplied by certain multiplicities, are together equal to the product of the degrees of S and f. (This appears e.g. by applying the theorem of BEZOUT to the intersection of the two surfaces with a plane chosen in a most general way).

An analagous theorem holds good in the general case. The intersection of M with the form f decomposes into irreducible manifolds; we shall only consider those which have exactly the dimension $r-1$ [1]). These may be found by seeking the primary components of the reduced dimension $r-1$ of the ideal (P, f). Let $Q_1, \dots, Q_s$ be these components, $M_1, \dots, M_s$ their manifolds. By the *multiplicity* of M_α as intersection of M and f we understand the length of the ideal Q; let l be this multiplicity. If further γ, δ are the degrees of f in x und y, *the relation*

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[1]) We can prove that these are the only ones. Cf. O. BLUMENTHAL, Math. Ann. 57, 1903), p. 356. An algebraic proof of this theorem was communicated to me by W. KRULL.

$$\sum_\alpha l_\alpha \, . \, a_{ij} (M_\alpha) = \gamma \, . \, a_{i+1,j} (M) + \delta \, . \, a_{i,j+1} (M) \quad . \quad . \quad . \quad . \quad (1)$$

is valid.

In order to prove this relation we replace the degrees of the manifolds by the degrees of the corresponding prime ideals. The relation becomes

$$\sum_\alpha l_\alpha \, . \, a_{ij} (P_\alpha) = \gamma \, . \, a_{i+1,j} (P) + \delta \, . \, a_{i,j+1} (P)$$

or

$$\sum_\alpha a_{ij} (Q_\alpha) = \gamma \, . \, a_{i+1,j} (P) + \delta \, . \, a_{i,j+1} (P)$$

or

$$a_{ij} (P, f) = \gamma \, . \, a_{i+1,j} (P) + \delta \, . \, a_{i,j+1} (P).$$

This is only a special case of theorem 10 (§ 20).

§ 35. We have defined the multiplicities of the partial intersections M as the lengths of certain primary ideals. This definition is only justified by its success: the sum of the products of degree and multiplicity is given by formula (1) and is, therefore, independent of the special situation of M_r and M_{n-r} relative to each other; in other words the definition of multiplicity satisfies the "rule of conservation of number". One would be mistaken by assuming that the notion of length always leads to a definition of multiplicity that satisfies this condition; on the contrary, already in the determination of the points of intersection of an M_2 with another M_2 in P_4 the notion length leads to a definition of multiplicity such that in certain cases where the two surfaces have the degrees 1 and 4 it depends on their relative position whether the sum of the "multiplicities" of their points of intersection is 4 or 5. In these cases we must reject the notion length and try to find another definition of multiplicity.[1]

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[1] Cf. my paper on "Eine Verallgemeinerung des Bezoutschen Satzes", Math. Ann. **99** (1928), p. 497.

Physics. — *On the structure of the spectrum of ionized Argon (Ar II).*
By T. L. DE BRUIN. (Communicated by Prof. P. ZEEMAN.)

(Communicated at the meeting of June 30, 1928).

1. *Introduction.*

In a former paper in these Proceedings [1]) I have communicated a partial analysis of the spectrum of ionized Argon (Ar II), in which especially the quartet system in connection with the analogous spectrum of ionized Neon (Ne II) is discussed. This paper contains further energy levels found from the spectrum and the theoretical identification. A list of all till now classified lines is given.

2. *Structure of the spectrum.*

The energy levels of the simply ionized Argon atom (Ar II) are built upon the ground levels of the double ionized atom (Ar III). These groundlevels arise from the atomic configuration of a 3-quantum orbit with two *3s* and four *3p* electrons. According to the theory of HEISENBERG, PAULI and HUND this configuration gives the lowest energy level as ^3P and also the metastable levels ^1D and ^1S. These terms are the basic terms of the ionized atom and according to the simple rules of HUND the following terms of Ar II can be deduced.

TABLE 1. Energy levels *Ar* II.

	Electronic configuration										Symbol	Basic term: ^3P		Basic term: ^1D	Basic term: ^1S
												Terms		Terms	Terms
1s	2s	2p	3s	3p	3d	4s	4p	4d	4f	5s		Quartet	Doublet	Doublet	Doublet
2	2	6	2	5							$s^2 p^5$		P		
2	2	6	2	4	1						$s^2 p^4 . 3d$	F D P	F D P	G F D P S	D
2	2	6	2	4		1					$s^2 p^4 . 4s$	P	P	D	S
2	2	6	2	4			1				$s^2 p^4 . 4p$	D P S	D P S	F D P	P
2	2	6	2	4				1			$s^2 p^4 . 4d$	F D P	F D P	G F D P S	D
2	2	6	2	4						1	$s^2 p^4 . 5s$	P	P	D	S

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[1]) DE BRUIN: These Proceedings **31**, N⁰. 6, 593, 1928. Zeitschr. f. Phys. **48**. 62. 1928.

The spectral lines arising from terms based upon the most stable state ^3P will appear as the strongest in the spectrum. The following figure gives a survey of the terms arising from this ^3P state through the coupling of a $3\,p$, $3\,d$, $4\,s$, $4\,p$, $5\,s$ and $4\,d$ electron.

For simplicity of the figure the fine structure of the levels (splitting according to the inner quantum numbers j) is not given. Each connecting line does not represent one spectralline, but a whole group, a multiplet.

For simplicity also the intercombinations are not given.

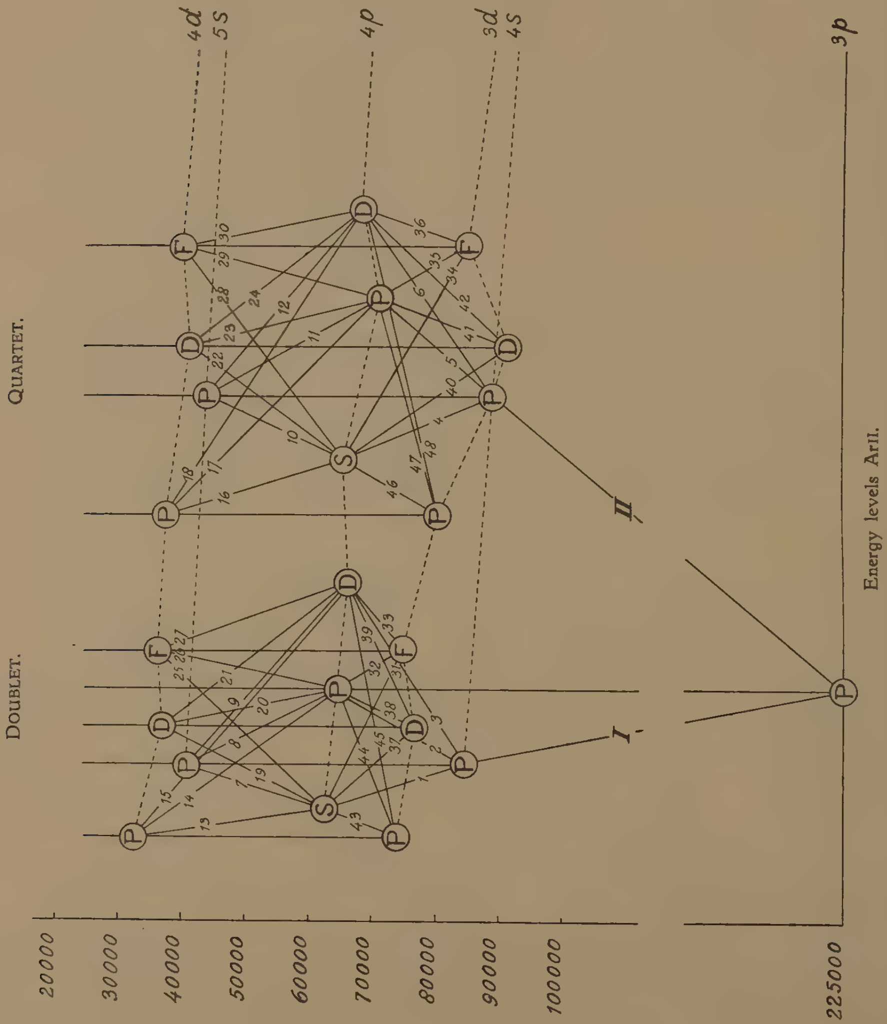

TABLE 2. Termtable Ar II (Basic term ^3P).

N⁰.	Exp. Term	Term-value	Term-difference	Theor. Term	N⁰.	Exp. Term	Term-value	Term-difference	Theor. Term
1	$^2P_2$	224758	1431	3 p $^2P_2$	36	$^2P_1$	65047.98	−532.96	4 p $^2P_1$
2	$^2P_1$	223327		3 p $^2P_1$	37	$^2P_2$	64515.02		4 p $^2P_2$
3	$^4D_4$	92427.28	153.98	3 d $^4D_4$	38	$^4D_4$	41079.08	121.80	4 d $^4D_4$
4	$^4D_3$	92273.30	149.62	3 d $^4D_3$	39	$^4D_3$	40957.28	188.61	4 d $^4D_3$
5	$^4D_2$	92123.68	107.03	3 d $^4D_2$	40	$^4D_2$	40768.67	206.29	4 d $^4D_2$
6	$^4D_1$	92016.65		3 d $^4D_1$	41	$^4D_1$	40562.38		4 d $^4D_1$
7	$^2D_3$	— 70000—82000		3 d $^2D_3$	42	$^2D_3$	40026.38	? 131.65	4 d $^2D_3$
8	$^2D_2$	—		3 d $^2D_2$	43	$^2D_2$	39894.73	?	4 d $^2D_2$
9	$^4F_5$	—		3 d $^4F_5$	44	$^4F_5$	39661.58	531.55	4 d $^4F_5$
10	$^4F_4$	—		3 d $^4F_4$	45	$^4F_4$	39130.03	449.59	4 d $^4F_4$
11	$^4F_3$	—		3 d $^4F_3$	46	$^4F_3$	38680.44	266.33	4 d $^4F_3$
12	$^4F_2$	—		3 d $^4F_2$	47	$^4F_2$	38414.11		4 d $^4F_2$
13	$^4P_1$	—		3 d $^4P_1$	48	$^4P_1$	38583.18	−299.00	4 d $^4P_1$
14	$^4P_2$	—		3 d $^4P_2$	49	$^4P_2$	38284.18	−420.60	4 d $^4P_2$
15	$^4P_3$	—		3 d $^4P_3$	50	$^4P_3$	37863.58		4 d $^4P_3$
16	$^2F_4$	—		3 d $^2F_4$	51	$^2F_4$	37938.38	772.50	4 d $^2F_4$
17	$^2F_3$	—		3 d $^2F_3$	52	$^2F_3$	37165.88	?	4 d $^2F_3$
18	$^2P_1$	—		3 d $^2P_1$	53	$^2P_1$	34819.88	−658.00	4 d $^2P_1$
19	$^2P_2$	—		3 d $^2P_2$	54	$^2P_2$	34161.88		4 d $^2P_2$
20	$^4P_3$	90512.88	844.40	4 s $^4P_3$	55	$^4P_3$	43160.38	627.76	5 s $^4P_3$
21	$^4P_2$	89668.48	515.70	4 s $^4P_2$	56	$^4P_2$	42532.62	729.14	5 s $^4P_2$
22	$^4P_1$	89152.78		4 s $^4P_1$	57	$^4P_1$	41803.48		5 s $^4P_1$
23	$^2P_2$	86510.88	1014.74	4 s $^2P_2$	58	$^2P_2$	41664.18	824.00	5 s $^2P_2$
24	$^2P_1$	85496.14		4 s $^2P_1$	59	$^2P_1$	40840.18		5 s $^2P_1$
25	$^4P_3$	69711.13	307.75	4 p $^4P_3$	60	$^4P_3$	— 30000 — 35000		5 p $^4P_3$
26	$^4P_2$	69403.38	357.30	4 p $^4P_2$	61	$^4P_2$	—		5 p $^4P_2$
27	$^4P_1$	69046.08		4 p $^4P_1$	62	$^4P_1$	—		5 p $^4P_1$
28	$^4D_4$	67520.58	439.36	4 p $^4D_4$	63	$^4D_4$	—		5 p $^4D_4$
29	$^4D_3$	67081.22	494.57	4 p $^4D_3$	64	$^4D_3$	—		5 p $^4D_3$
30	$^4D_2$	66586.65	260.32	4 p $^4D_2$	65	$^4D_2$	—		5 p $^4D_2$
31	$^4D_1$	66326.33		4 p $^4D_1$	66	$^4D_1$	—		5 p $^4D_1$
32	$^2D_3$	66024.28	663.90	4 p $^2D_3$	67	$^2D_3$	—		5 p $^2D_3$
33	$^2D_2$	65360.38		4 p $^2D_2$	68	$^2D_2$	—		5 p $^2D_2$
34	$^4S_2$	63705.60		4 p $^4S_2$	69	$^4S_2$	—		5 p $^4S_2$
35	$^2S_1$	63665.08		4 p $^2S_1$	70	$^2S_1$	—		5 p $^2S_1$

3. *Termtable. Ionization potential.*

The following energy levels are found from the analysis of the spectrum (See table 2).

As the figure shows some Rydberg series are found. These can serve us for calculating the absolute term values of the spectrum. For that purpose the serie $4\,s\,{}^4P_2$ and $5\,s\,{}^4P_2$ is the best one, because the variation of the quantum defect will be a minimum here. These two terms can be represented by:

$$4\,s\,{}^4P_2 = 89668{,}48 = \frac{4\,R}{(4 - 1{,}787483)^2}$$

and

$$5\,s\,{}^4P_2 = 42532{,}62 = \frac{4\,R}{(5 - 1{,}787483)^2}.$$

This gives a value of 224758 frequency units for the groundstate of the ionized Argon atom, which agrees with an ionization potential of 27.75 ± 0.05 Volt. Because the ionization potential of Ar I is 15.69 Volt [1]) one needs 15.69 + 27.75 Volt = 43.44 Volt to produce the whole spectrum from the ground state of the neutral atom. This agrees rather well with the value experimentally found by BARTON: 45.3 ± 1.5 Volt [2]). BARTON got this value by means of the investigation of $\frac{e}{m}$ for ions, using the method of SMYTH [3]).

4. *Experimental part.*

I have made new experiments by using a grating mounting (grating 8 × 5 cm total number of lines 51949) to check the consisting wavelength-measurements [4]). The spectrum has been produced in the same manner as in the case of ionized Neon [5]), we made use of a heavy condensed discharge through a narrow capilar ($^3/_4$ mm) [6]). The analysis of the spectrum had already shown, that measurement of the complete spectrum was not necessary and that the accurate wavelength measurements of KAYSER could be used. I have however found by means of a discharge of high brilliancy new, mostly weak lines. The measurements of the author are given in hundredths of Angström, those of KAYSER in thousandths of Angström.

The spectrum of the double ionized atom (Ar III) could also be produced

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[1]) MEISSNER: Zeitschr. f. Phys. **37**. 238. 1926; **39**. 172. 1926; **40**. 839. 1927.

[2]) BARTON: Phys. Rev. **25**. 469. 1925.

[3]) SMYTH: Phys. Rev. **25**. 452. 1925.

[4]) KAYSER: Handbuch. V. **61**; EDER und VALENTA: Denkschr. Wien. Akad. **64**. I. 1897. Beitr. z. Photochem. 1904; L. BLOCH, E. BLOCH et G. DÉJARDIN: Ann. de Phys. 2. 461. 1924.

[5]) DE BRUIN: Zeitschr. f. Phys. **44**. 157. 1927; **46**. 856. 1928.

[6]) I express my thanks to Mr. J. V. D. ZWAAL, chief of the technical service of the laboratory, for making the quartz tubes.

5. *Classification of the Ar II lines* [1]).

TABLE 3. Classification of the Ar II lines.

Int.	λ I. A.	ν_{vacuum}	Termcombination	M [2])	Int.	λ I. A.	ν_{vacuum}	Termcombination	M
1	5287.00	18909.0	4 s $^2P_1$ — 4 p $^4D_2$	—	7	4430.185	22566.11	4 s $^4P_1$ — 4 p $^4D_2$	6
3	5145.395	19429.45	4 s $^2P_2$ — 4 p $^4D_3$	—	10	4425.995	22587.23	4 s $^4P_2$ — 4 p $^4D_3$	6
5	5062.019	19749.47	4 s $^4P_1$ — 4 p $^4P_2$	5	4	4420.943	22613.14	3 d $^4D_1$ — 4 p $^4P_2$	41
4	5017.151	19926.08	4 s $^2P_2$ — 4 p $^4D_2$	—	7	4400.986	22715.81	3 d $^4D_4$ — 4 s $^4P_3$	41
5	5009.246	19957.53	4 s $^4P_2$ — 4 p $^4P_3$	5	6	4400.101	22720.39	3 d $^4D_2$ — 4 s $^4P_2$	41
3	4972.16	20106.4	4 s $^4P_1$ — 4 p $^4P_1$	5	2	4383.730	22805.24	4 s $^2P_2$ — 4 p $^4S_2$	—
6	4965.059	20135.14	4 s $^2P_1$ — 4 p $^2D_2$	3	8	4379.657	22826.44	4 s $^4P_1$ — 4 p $^4D_1$ 4 p $^2S_1$ — 5 s $^2P_1$	6 7
5	4933.226	20265.07	4 s $^4P_2$ — 4 p $^4P_2$	5	5	4375.942	22845.82	4 s $^2P_2$ — 4 p $^2S_1$	1
3	4889.06	20448.1	4 s $^2P_1$ — 4 p $^2P_1$	2	2	4375.031	22850.68	4 p $^2P_2$ — 5 s $^2P_2$	8
8	4879.824	20486.83	4 s $^2P_2$ — 4 p $^2D_3$	3	5	4371.334	22869.90	3 d $^4D_3$ — 4 p $^4P_2$	41
2	4866.00	20545.0	4 p $^4S_2$ — 5 s $^4P_3$	10	1	4358.48	22937.3	4 p $^4S_2$ — 4 d $^4D_2$	22
6	4847.783	20622.25	4 s $^4P_2$ — 4 p $^4P_1$	5	5	4352.198	22970.46	3 d $^4D_1$ — 4 p $^4P_1$	41
9	4805.993	20801.56	4 s $^4P_3$ — 4 p $^4P_3$	5	10	4348.062	22992.30	4 s $^4P_3$ — 4 p $^4D_4$	6
5	4764.848	20981.18	4 s $^2P_1$ — 4 p $^2P_2$	2	5	4332.035	23077.37	3 d $^4D_2$ — 4 s $^4P_1$	41
7	4735.885	21109.49	4 s $^4P_3$ — 4 p $^4P_2$	5	7	4331.194	23081.85	4 s $^4P_2$ — 4 p $^4D_2$	6
6	4726.847	21149.85	4 s $^2P_2$ — 4 p $^2D_2$	3	1	4319.67	23143.4	4 d $^4D_1$ — 4 p $^4S_2$	22
3	4721.66	21173.1	4 p $^4S_2$ — 5 s $^4P_2$	10	5	4282.894	23342.16	4 s $^4P_2$ — 4 p $^4D_1$	6
6	4657.889	21462.96	4 s $^2P_2$ — 4 p $^2P_1$	2	4	4275.167	23384.35	4 p $^2P_1$ — 5 s $^2P_2$	8
1	4588.00	21789.7	4 p $^4S_2$ — 4 s $^2P_1$	—	3	4267.49	23426.4	4 p $^4D_2$ — 5 s $^4P_3$	12
6	4579.347	21831.08	4 s $^2P_1$ — 4 p $^2S_1$	1	6	4266.524	23431.71	4 s $^4P_3$ — 4 p $^4D_3$	6
1	4564.50	21902.1	4 p $^4S_2$ — 5 s $^4P_1$	10	2	4255.57	23492.0	4 p $^2D_3$ — 5 s $^4P_2$	—
2	4547.79	21982.5	4 p $^2P_2$ — 5 s $^4P_2$	—	5	4228.150	23644.37	4 s $^4P_2$ — 4 p $^2D_3$	—
7	4545.040	21995.86	4 s $^2P_2$ — 4 p $^2P_2$	2	6	4222.679	23675.00	4 p $^2P_2$ — 5 s $^2P_1$	8
5	4502.941	22201.51	4 p $^2D_2$ — 5 s $^4P_3$	—	5	4218.683	23697.43	4 p $^2D_2$ — 5 s $^2P_2$	9
2	4535.57	22041.7	4 p $^4S_2$ — 5 s $^2P_2$	—	3	4201.946	23791.82	4 s $^4P_1$ — 4 p $^2D_2$	12
4	4460.512	22412.69	3 d $^4D_2$ — 4 p $^4P_3$	41	4	4179.319	23920.62	4 p $^4D_3$ — 5 s $^4P_3$	12
5	4431.002	22561.95	3 d $^4D_3$ — 4 p $^4P_3$	41	3	4178.344	23926.21	4 s $^4P_3$ — 4 p $^4D_2$	6

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[1]) For the connection between classification and ZEEMAN-effect we refer to: C. J. BAKKER, T. L. DE BRUIN and P. ZEEMAN, these Proceedings **31**, N⁰. 7, 780, 1928.

[2]) The column M refers to the numbers of the corresponding multiplets in the figure.

TABLE 3 (*Continued*).

Int.	λ I. A.	ν_{vacuum}	Termcombination	M
1	4173.77	23952.4	4 p $^2P_2$ — 4 d $^4D_1$	—
6	4156.135	24054.06	4 p $^4D_2$ — 5 s $^4P_2$	12
1	4129.67	24208.2	4 p $^2P_1$ — 5 s $^2P_1$	8
4	4112.766	24307.70	4 s $^4P_2$ — 4 p $^2D_2$	—
10	4103.957	24359.88	4p $^4D_4$ — 5 s $^4P_3$ 4p $^2D_3$ — 5 s $^2P_2$	12 9
5	4082.385	24488.59	4p $^2P_2$ — 4 d $^2D_3$ 4s $^4P_3$ — 4 p $^2D_3$	20 —
2	4077.057	24520.59	5 s $^2P_1$ — 4 p $^2D_2$	9
6	4076.704	24522.72	4 p $^4D_1$ — 5 s $^4P_1$	12
4	4072.429	24548.48	4 p $^4D_3$ — 5 s $^4P_2$	12
1	4065.25	24591.8	4 p $^2D_2$ — 4 d $^4D_2$	—
4	4038.816	24752.76	3 d $^4D_3$ — 4 p $^4D_4$	42
5	4033.872	24783.10	4 p $^4D_2$ — 5 s $^4P_1$	12
1	4031.47	24797.8	4 p $^2D_2$ — 4 d $^4D_1$	—
8	4013.852	24906.71	3 d $^4D_4$ — 4 p $^4D_4$	42
1	4011.377	24922.08	4 p $^4D_2$ — 5 s $^2P_2$	—
1	3994.885	25025.16	4 p $^4S_2$ — 4 d $^4F_3$	28
4	3992.046	25042.75	3 d $^4D_2$ — 4 p $^4D_3$	42
2	3988.228	25066.73	4 p $^2D_3$ — 4 d $^4D_3$	—
6	3979.391	25122.39	4 p $^4S_2$ — 4 d $^4P_1$	16
3	3974.709	25151.98	4 s $^4P_3$ — 4 p $^2D_2$	—
3	3974.496	25153.33	4s $^4P_2$ — 4 p $^2P_2$ 4p $^2P_1$ — 4 d $^2D_2$	— 20
5	3968.346	25192.32	3 d $^4D_3$ — 4 p $^4D_3$	42
2	3958.379	25255.75	4 p $^2D_3$ — 4 d $^4D_2$	—
3	3952.742	25291.76	4 p $^4S_2$ — 4 d $^4F_2$	28
6	3946.140	25334.07	4 p $^2D_2$ — 4 d $^2D_3$	21
4	3944.259	25346.16	3 d $^4D_4$ — 4 p $^4D_3$	42
6	3932.567	25421.51	4 p $^4S_2$ — 4 d $^4P_2$	16
3	3931.232	25430.14	3 d $^4D_1$ — 4 p $^4D_2$	42
6	3928.599	25447.20	4 s $^4P_1$ — 4 p $^4S_2$	4
5	3925.753	25465.65	4 p $^2D_2$ — 4 d $^2D_2$	21
4	3914.781	25537.01	3 d $^4D_2$ — 4 p $^4D_2$	42
2	3911.571	25557.96	4 p $^4D_1$ — 4 d $^4D_2$	24
3	3900.613	25629.76	4 p $^4D_2$ — 4 d $^4D_3$	24
4	3891.978	25686.62	3 d $^4D_3$ — 4 p $^4D_2$	42
3	3891.400	25690.44	3 d $^4D_1$ — 4 p $^4D_1$	42
3	3880.281	25764.04	4 p $^4D_1$ — 4 d $^4D_1$	24
4	3875.256	25797.49	3 d $^4D_2$ — 4 p $^4D_1$	42
3	3872.176	25817.98	4 p $^4D_2$ — 4 d $^4D_2$	24
8	3868.568	25842.06	4 p $^4S_2$ — 4 d $^4P_3$	16
9	3850.565	25962.88	4 s $^4P_2$ — 4 p $^4S_2$	4
3	3845.385	25997.86	4p $^2D_3$ — 4 d $^2D_3$ 4s $^4P_3$ — 4 p $^2P_2$	21 —
3	3844.771	26002.00	4 p $^4D_3$ — 4 d $^4D_4$	24
4	3841.559	26023.74	4 p $^4D_2$ — 4 d $^4D_1$	24
3	3830.435	26099.31	3 d $^4D_2$ — 4 p $^2D_2$	—
5	3826.826	26123.93	4 p $^4D_3$ — 4 d $^4D_3$	24
3	3825.715	26131.52	4 p $^2D_3$ — 4 d $^2D_2$	21
5	3809.499	26242.74	4 p $^4P_2$ — 5 s $^4P_3$	11
3	3808.596	26248.97	3 d $^4D_3$ — 4 p $^2D_3$	—
4	3799.446	26312.18	4 p $^4D_3$ — 4 d $^4D_2$	24
3	3786.386	26402.93	3 d $^4D_4$ — 4 p $^2D_3$	—
7	3780.868	26441.47	4 p $^4D_4$ — 4 d $^4D_4$	24
4	3770.569	26513.69	4 p $^4P_1$ — 5 s $^4P_2$	11
6	3765.313	26550.70	4 p $^4P_3$ — 5 s $^4P_3$	11
5	3763.565	26563.03	4 p $^4D_4$ — 4 d $^4D_3$	24
3	3750.278	26657.14	3 d $^4D_1$ — 4 p $^2D_2$	—
2	3746.985	26680.57	4 p $^2D_2$ — 4 d $^4F_3$	—
1	3735.392	26763.37	3 d $^4D_2$ — 4 p $^2D_2$	—
10	3729.300	26807.09	4 s $^4P_3$ – 4 p $^4S_2$	4
3	3720.467	26870.73	4 p $^4P_2$ — 5 s $^4P_2$	11
3	3717.217	26894.22	4 p $^2D_3$ — 4 d $^4F_4$	—
1	3714.694	26913.22	3 d $^4D_3$ — 4 p $^2D_2$	—
2	3710.017	26946.41	4 p $^2D_2$ — 4 d $^4F_2$	—

T. L. DE BRUIN: ON THE STRUCTURE OF THE SPECTRUM OF IONIZED ARGON (AR II).

ARGON II.

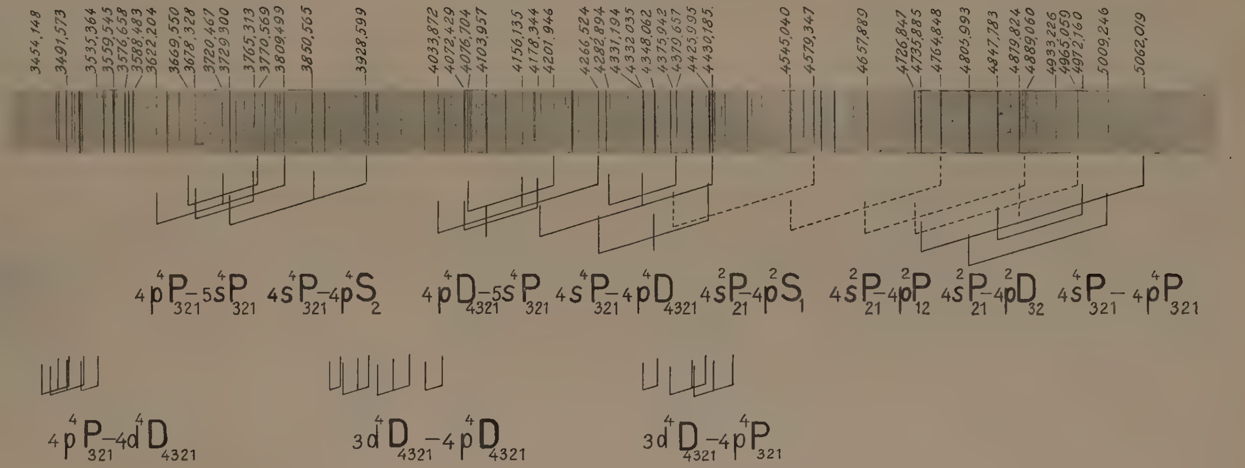

TABLE 3 (*Continued*).

Int.	λ I. A.	ν_{vacuum}	Termcombination	M	Int.	λ I. A.	ν_{vacuum}	Termcombination	M
3	3678.328	27178.54	$4p\ {}^4P_3 - 5s\ {}^4P_2$	11	8	3480.486	28723.43	$4p\ {}^4D_2 - 4d\ {}^4P_3$	18
1	3669.550	27243.56	$4p\ {}^4P_1 - 5s\ {}^4P_1$	11	6	3476.776	28754.07	$4p\ {}^4P_3 - 4d\ {}^4D_3$	23
3	3656.120	27343.64	$4p\ {}^2D_3 - 4d\ {}^4F_3$	—	4	3466.383	28840.29	$4p\ {}^4P_2 - 4d\ {}^4D_1$ $4p\ {}^4D_4 - 4d\ {}^4F_3$	23 30
3	3650.991	27382.05	$4p\ {}^4P_1 - 5s\ {}^2P_2$	—	5	3454.148	28942.44	$4p\ {}^4P_3 - 4d\ {}^4D_2$	23
4	3622.204	27599.69	$4p\ {}^4P_2 - 5s\ {}^4P_1$	11	3	3430.400	29142.79	$4p\ {}^4D_3 - 4d\ {}^2F_4$	—
1	3603.97	27739.3	$4s\ {}^4P_2 - 5s\ {}^2P_2$	—	2	3421.671	29217.14	$4p\ {}^4D_3 - 4d\ {}^4P_3$	18
1	3603.53	27742.7	$4p\ {}^2D_3 - 4d\ {}^2P_2$	15	3	3388.566	29502.57	$4p\ {}^2S_1 - 4d\ {}^2P_2$	13
10	3588.483	27859.00	$4p\ {}^4D_4 - 4d\ {}^4F_5$	30	1	3379.424	29582.07	$4p\ {}^4D_4 - 4d\ {}^2F_4$	—
7	3582.397	27906.23	$4p\ {}^4D_2 - 4d\ {}^4F_3$	30	3	3370.937	29656.85	$4p\ {}^4D_4 - 4d\ {}^4P_3$	18
6	3581.652	27912.14	$4p\ {}^4D_1 - 4d\ {}^4F_2$	30	2	3366.618	29694.90	$4p\ {}^2P_2 - 4d\ {}^2P_1$	14
10	3576.658	27951.11	$4p\ {}^4D_3 - 4d\ {}^4F_4$	30	2	3307.228	30228.31	$4p\ {}^2P_1 - 4d\ {}^2P_1$	14
2	3565.071	28041.95	$4p\ {}^4D_1 - 4d\ {}^4P_2$	18	5	3293.628	30353.05	$4p\ {}^2P_2 - 4d\ {}^2P_2$	14
1	3564.436	28046.94	$4p\ {}^4P_1 - 5s\ {}^2P_2$	—	3	3281.727	30463.01	$4p\ {}^4P_1 - 4d\ {}^4P_1$	17
5	3559.545	28085.49	$4p\ {}^2D_3 - 4d\ {}^2F_4$	27	2	3273.336	30541.09	$4p\ {}^2D_2 - 4d\ {}^2P_1$	15
5	3548.530	28172.66	$4p\ {}^4D_2 - 4d\ {}^4F_2$	30	4	3263.582	30632.37	$4p\ {}^4P_1 - 4d\ {}^4F_2$	29
8	3545.642	28195.59	$4p\ {}^2D_2 - 4d\ {}^2F_3$	27	5	3249.842	30761.88	$4p\ {}^4P_1 - 4d\ {}^4P_2$	17
5	3535.364	28277.57	$4p\ {}^4P_1 - 4d\ {}^4D_2$	23	5	3243.705	30820.08	$4p\ {}^4P_2 - 4d\ {}^4P_1$	17
3	3521.281	28390.66	$4p\ {}^4D_4 - 4d\ {}^4F_4$	30	1	3236.682	30886.94	$4p\ {}^2P_1 - 4d\ {}^2P_2$	14
4	3520.041	28400.66	$4p\ {}^4D_3 - 4d\ {}^4F_3$	30	1	3226.035	30988.94	$4p\ {}^4P_2 - 4d\ {}^4F_2$	29
1	3517.929	28417.71	$3d\ {}^4D_2 - 4p\ {}^4S_2$	40	1	3212.597	31118.50	$4p\ {}^4P_2 - 4d\ {}^4P_2$	17
8	3514.426	28446.04	$4p\ {}^4P_2 - 4d\ {}^4D_3$	23	3	3204.339	31198.70	$4p\ {}^2D_2 - 4d\ {}^2P_2$	14
3	3509.811	28483.44	$4p\ {}^4P_1 - 4d\ {}^4D_1$	23	2	3194.270	31297.04	$4p\ {}^4F_3 - 4d\ {}^4F_2$	29
1	3499.665	28566.02	$3d\ {}^4D_3 - 4p\ {}^4S_2$	40	4	3181.044	31427.16	$4p\ {}^4P_3 - 4d\ {}^4P_2$	17
9	3491.573	28632.22	$4p\ {}^4P_3 - 4d\ {}^4D_4$	23	5	3169.682	31539.81	$4p\ {}^4P_2 - 4d\ {}^4P_3$	17
5	3491.290	28634.54	$4p\ {}^4P_2 - 4d\ {}^4D_2$	23	6	3139.026	31847.82	$4p\ {}^4P_3 - 4d\ {}^4P_3$	17

in this manner. The measurement and the analysis of this spectrum has already been begun.

The plate shows a reproduction of a part of the spectrum. Several important multiplets in this region are marked, while for orientation several wavelengths are added.

6. *The spectrum in the extreme violet* [1]).

On account of the analysis of the visible Ar II spectrum an interpretation can be given of the line groups found by DORGELO and ABBINK [2]) in the extreme violet. Those are combinations with the deep groundstate $3\,p\;{}^2P_{21}$:

I.	4.	730,91	136816	$3p\;{}^2P_1 - 4s\;{}^2P_2$
	3.	725,53	137830	$3p\;{}^2P_1 - 4s\;{}^2P_1$
	3.	723,35	138246	$3p\;{}^2P_2 - 4s\;{}^2P_2$
	1.	718,05	139266	$3p\;{}^2P_2 - 4s\;{}^2P_1$
II.	6.	740,23	135093	$3p\;{}^2P_2 - 4s\;{}^4P_2$
III.	12.	932,06	107289	$3p\;{}^2P_1 - s\,p^6\;{}^2S_1$
	15.	919,79	108720	$3p\;{}^2P_2 - s\,p^6\;{}^2S_1$

Some of the other groups found belong to the spectrum of the double ionized atom (Ar III).

7. *Comparison with other atomic configurations.*

The analysis of a spectrum is especially important for the study of the binding of electrons by means of comparison with other atomic configurations. The iso-electron configurations of Cl I, Ar II, K III cannot yet be compared, because an analysis of Cl I and K III fails.

We can now however compare the atomic configurations :

17. Cl II, 18. Ar II, 19. K II, 20. Ca. II, 21. Sc. II, because in the last time something more is known on the spectra of Cl II [3]), K II [4]) and Sc II [5]). We will not give an extensive comparison but only some examples.

a. The ionization potentials in the above mentioned configurations are (25) ; 27.75 ; 31.68 ; 11.81 and 12.84 Volt. The discontinual decay in the case of Ca II is clear, because in the case of Cl II, Ar II and K II a 4 *s*-electron less strongly bound must be removed.

b. We now give attention to the energy values of the 4 *s*-orbits in the above mentioned configurations. Because the *s* orbits are orbits with inner loop which means they partially come into the atomic core in the neighbourhood of the nucleus, one will expect with increasing nuclear charge stronger binding, thus higher energy value and larger quantum defect, as also is exhibited in the following table.

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[1]) Prof. K. T. COMPTON had the kindness to send me the copy of the paper on the spectrum of *Ar II* in the extreme violet (K. T. COMPTON, J. C. BOYCE and H. N. RUSSELL), which will be printed in the "Physical Review". For the classification of the *Ar II* lines in the extreme violet we refer to that paper.

[2]) DORGELO und ABBINK: Zeitschr. f. Phys. **41**. 753. 1927.

[3]) BOWEN: Phys. Rev. **31**. 34. 1928.

[4]) DE BRUIN: Archives Néerl. **11**. 70. 1928; BOWEN: Phys. Rev. **31**. 497. 1928.

[5]) RUSSELL and MEGGERS: Sc. P. Bur. of. Stand. N⁰. 558. 329. 1927.

Atomic-configuration	Energyvalue 4 s-terms	Effective principal quantum number	Quantum-defect
17 . Cl II	84000	2.28	1.72
18 . A II	88000	2.23	1.77
19 . K II	92000	2.18	1.82
20 . Ca II	96000	2.14	1.86
21 . Sc II	102000	2.07	1.93

8. *Summary.*

The spectrum of ionized Argon has been investigated experimentally. Some 180 lines have been ordered in a term scheme. From the 59 theoretical levels predicted by HUND's theory, arising from the coupling 3 *p*, 3 *d*, 4 *s*, 4 *p*, 4 *d*, 5 *s* (^3P), 46 have been identified. The remaining terms lay in regions which, combining, give spectral lines in the infra red and the extreme violet.

The ionization potential has been found from the spectrum and amounts to 27.75 ± 0.05 Volt. A comparison is made with other, in the periodic system neighbouring atomic configurations.

In conclusion the author wishes to express thanks to Prof. ZEEMAN for valuable advice and helpful suggestions.

Laboratory "Physica" of the University
of Amsterdam.

June; 1928.

Physics. — *The* ZEEMAN-*effect of the spectrum of ionized Argon (Ar II).* By C. J. BAKKER, T. L. DE BRUIN and P. ZEEMAN.

(Communicated at the meeting of June 30, 1928.)

1. *Introduction.*

The ZEEMAN-effect of the spectrum of ionized Argon *(Ar II)* has been investigated for checking the analysis of this spectrum given by one of us (DE BRUIN) [1]. The research [2] was especially important because one can expect in this spectrum according to theoretical considerations deviations from the normal *g*-values of LANDÉ. The theory of BOHR gives that the atomic core of *Ar II* contains four *3p*-electrons. According to the conception of LANDÉ this should give rise to anomalous *g*-values. Certain groups of *g*-values must give, on account of theoretical considerations of PAULI, LANDÉ and HEISENBERG constant "*g*-sums", which can be calculated beforehand.

2. *Experimental part.*

The grating mounting.

We made use of the stigmatic grating mounting of the laboratory. The light from the source, whose image is focussed on the slit, falls upon a concave mirror, which sends a parallel beam of light to the grating [3]. We could photograph at once a region of 15 cm at both sides of the normal of the grating. The investigation was made in the second order and there the dispersion is about 2.4 Å/mm, so we could photograph with one focussing of the camera a region of 300 × 2.4 Å = 720 Å; thus we had to focuss four times for investigating the spectral region between 2700 and 5300 Å.

The 5 inch grating used has 14438 lines to the inch (originally the width of the grating is 6 inches, but one inch gives disturbances and was therefore covered during the exposures). For resolving very narrow magnetic separations it is necessary that the temperature of the grating remains constant to at least 0°.05 during a whole exposure. Especially for this purpose an automatical regulation of the temperature is made in the large room of the laboratory where the grating is mounted [1]). This regulation enables us to keep the temperature of the grating constant to 0°.01 during the exposure.

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[1] DE BRUIN:. These Proceedings **31**, N⁰. 6, 593, 1928 and **31**, N⁰. 7, 771, 1928. Zeitschr. f. Phys., **48**, 62, 1928.

[2] The ZEEMAN-effect is not very much investigated. Some lines in the visible part of the spectrum has been investigated by LÜTTIG (Dissertatie Halle a. S. 1911; Ann. der Phys. **38**, 69, (1912).

[3] RUNGE und PASCHEN, Wied. Ann., **61**, 641, 1897. P. ZEEMAN used this mounting since 1900 for many investigations.

MEGGERS and KEIVIN BURNS, Sc. Pap. Bur. of Stand. N⁰. 441, (vol. 18), 1922.

The magnetic field.

The magnet is a large WEISS-electromagnet, with water cooled copper coils, made by the engine factory „Oerlikon" (Zürich). It is operated on a current of 100 Ampères. The field is then with the used distance of the endplanes of the poles about 41000 Gauss. The cooling is very effective and makes that the temperature of the magnet does not change observably; in this way satisfactory constancy of the field is obtained.

The light source.

We made use of a gasdischarge *without* Geissler tube, that passes parallel to the field. The Argon gas is therefore led into a copper "BACK-box" [2]) enclosing the magnetic poles and provided with quartz windows. The gas discharge passes between 2 circular platin discs, just covering the pole tips of the magnet (5 mm in diameter). The platin discs are insulated from the pole pieces by using thin pieces of mica. Platin discs and mica are attached to conical plugs, to our purpose made of ebonite and ground to fit into openings into the box. *Without* the magnetic field the discharge spreads through the whole of vacuum space, *with* magnetic field and well choosen gas pressure (in the case of Argon between 2 and 3 cm) the discharge contracts to the central part of the field and comes to high brilliancy there. We used un uncondensed high potential alternating current furnished by a large transformer that operated indefinitely on ½ K.W., while the current in the secundary was 25 m.A.

The times of exposure varied between 4 and 6 hours.

Intensity measurement of the field.

The intensity of the magnetic field can be measured by means of the exactly known ZEEMAN-effect of spectral lines, that may not show dissymmetries in their magnetic separations. In connexion with the spectral region that was investigated, we used the ZEEMAN-effect of the well known *Zn*-triplet 4811, 4722 and 4680 or of the *Ag* grounddoublet 3883 and 3281.

When none of these lines fell in the spectral region we investigated, we could satisfactory make use of the accurately known and measured ZEEMAN-effect of *Ar II* lines, because the regions we investigated always covered each other over some distance.

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[1]) We refer to: GEHRCKE: Handb. der Physik. Optik, Band II, zweite Hälfte, erster Teil. P. ZEEMAN und T. L. DE BRUIN: Magnetische Zerlegung der Spektrallinien, p. 605.

[2]). Struktur der Materie I: E. BACK und A. LANDÉ: ZEEMAN-effekt und Multiplettstruktur der Spektrallinien.

	λ	Termcombination	ZEEMAN-effect
Zn.	4810.534	$^3P_2 - {}^3S_1$	**(0.00)** (0.50) **1.00** 1.50 2.00
	4722.163	$^3P_1 - {}^3S_1$	(0.50) 1.50 2.00
	4680.140	$^3P_0 - {}^3S_1$	(0.00) 2.00
Ag	3382.89	$^2S_1 - {}^2P_1$	(0.67) 1.33
	3280.67	$^2S_1 - {}^2P_2$	**(0.33)** **1.00** 1.67

3. *Normal and anomalous g-values.*

According to the theory of LANDÉ[1]) the ZEEMAN-effect of a classified spectralline, arising from the combination between two energy levels (terms) can be calculated. An external magnetic field splits up a term with the inner quantumnumber j in $2j+1$ sublevels with the magnetic quantumnumbers $m = j,\ j-1, \ldots -j$. The energy alteration of the term, $\triangle E = h \,.\, \triangle \nu$, is given bij the expression: $\triangle \nu = m.\, g.\, o.$ $\left(o = \frac{e}{m} \cdot \frac{H}{4\pi c} \right.$ means the normal resolution$\left.\right)$ g is LANDÉ's splitting factor, which can be calculated directly by means of the quantumnumbers of a term namely;

$$g = 1 + \frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)},$$

in which j = inner quantumnumber, s = rotational quantumnumber, l = azimuthal quantumnumber.

EXAMPLE: A II. $\lambda\, 3729{,}300 = 4s\, {}^4P_3 - 4p\, {}^4S_2$.[2])

$4s\, {}^4P_3 : j = {}^5/_2,\ s = {}^3/_2,\ l = 1,\ g = 1{,}60;\ 2j+1 = 6$ magnetic levels with $m = {}^5/_2,\ {}^3/_2,\ {}^1/_2,\ -{}^1/_2,\ -{}^3/_2,\ -{}^5/_2.$

$4p\, {}^4S_2 : j = {}^3/_2,\ s = {}^3/_2,\ l = 0,\ g = 2{,}00;\ 2j+1 = 4$ magnetic levels with $m = {}^3/_2,\ {}^1/_2,\ -{}^1/_2,\ -{}^3/_2.$

The levels with $\triangle m = 0$ and ± 1 combine ($\triangle m = 0$, π-component, $\triangle m = \pm 1$, σ-component).

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[1]) Struktur der Materie I; E. BACK und A. LANDÉ: ZEEMAN-effekt und Multiplettstruktur der Spektrallinien.

[2]) The inner quantum number j is for typographical simplicity replaced by the whole number: $j + {}^1/_2$.

Thus the calculated energy alterations are:

$$\begin{array}{lllllll} 4s\,^4P_3: & -4{,}00 & -2{,}40 & -0{,}80 & 0{,}80 & 2{,}40 & 4{,}00 \\ 4p\,^4S_2: & & -3{,}00 & -1{,}00 & 1{,}00 & 3{,}00 & \end{array}$$

combine

$$-2{,}20 \quad -1{,}80 \quad -1{,}40 \quad -\mathbf{1{,}00} \quad -(0{,}60) \quad -(\mathbf{0{,}20}) \quad (\mathbf{0{,}20}) \quad (0{,}60) \quad \mathbf{1{,}00} \quad 1{,}40, \quad 1{,}80 \quad 2{,}20$$

or:

$$\pm(\mathbf{0{,}20}) \quad (0{,}60) \quad \mathbf{1{,}00} \quad 1{,}40 \quad 1{,}80 \quad 2{,}20.$$

Observed is:

$$\pm(\mathbf{0{,}20}) \quad (0{,}60) \quad \mathbf{1{,}00} \quad 1{,}40 \quad 1{,}80 \quad 2{,}20.$$

It appears however that in many cases the ZEEMAN-effect calculated in this way does not agree with the experiments. One supposes, that when the g-formula holds, the coupling of the quantumnumbers is the normal one, (RUSSELL-SAUNDERS coupling) symbolically given by

$$\{[(s_1\,s_2\ldots)\,(l_1\,l_2\ldots)]\,H\} = (s\,l)\,H = jH.$$

Other kinds of coupling give rise to anomalous $g =$ values [1]).

EXAMPLE: Multiplet 2.

A II. $\lambda\,4657{,}889 = 4s\,^2P_2 - 4p\,^2P_1$. According to LANDÉ's formula the normal g-values are $g\,^2P_2 = 1{,}33$ and $g\,^2P_1 = 0{,}67$ and the ZEEMAN-effect calculated with these g-values is

$$\mathbf{1{,}67} \quad 1{,}00 \quad (\mathbf{0{,}33}) \quad (\mathbf{0{,}33}) \quad 1{,}00 \quad \mathbf{1{,}67}.$$

Observed however is:

$$\mathbf{1{,}50^5} \quad 1{,}14^5 \quad (\mathbf{0{,}16^5}) \quad (\mathbf{0{,}16^5}) \quad 1{,}21 \quad \mathbf{1{,}51},$$

From this are now calculated the g-values $g_1 = g\;4s\,^2P_2$ and $g_2 = g\;4p\,^2P_1$ as follows:

$$\begin{array}{lllll} 4s\,^2P_2: & -{}^3/_2\,g_1 & -{}^1/_2\,g_1 & +{}^1/_2\,g_1 & +{}^3/_2\,g_1 \\ 4p\,^2P_1: & & -{}^1/_2\,g_2 & +{}^1/_2\,g_2 & \end{array}$$

combine

$$\pm\left({}^1/_2\,(g_1 - g_2)\right) \quad {}^1/_2\,(g_1 + g_2) \quad {}^3/_2\,g_1 - {}^1/_2\,g_2$$

This gives:

$${}^1/_2\,(g_1 - g_2) = 0{,}16^5 \text{ and } {}^3/_2\,g_1 - {}^1/_2\,g_2 = 1{,}50^5$$

thus:

$$g_1 = 1{,}33 \text{ and } g_2 = 0{,}99.$$

Hence we see that $g\;4p\,^2P_1$ is an anomalous one.

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[1]) S. GOUDSMIT und G. E. UHLENBECK: Zeitschr. f. Phys. **35**, 618, 1926.
S. GOUDSMIT und E. BACK: Zeitschr. f. Phys. **40**, 530. 1927.

The observed ZEEMAN-effect of the line λ 4764,848 $= 4s\,^2P_1 - 4p\,^2P_2$ is:

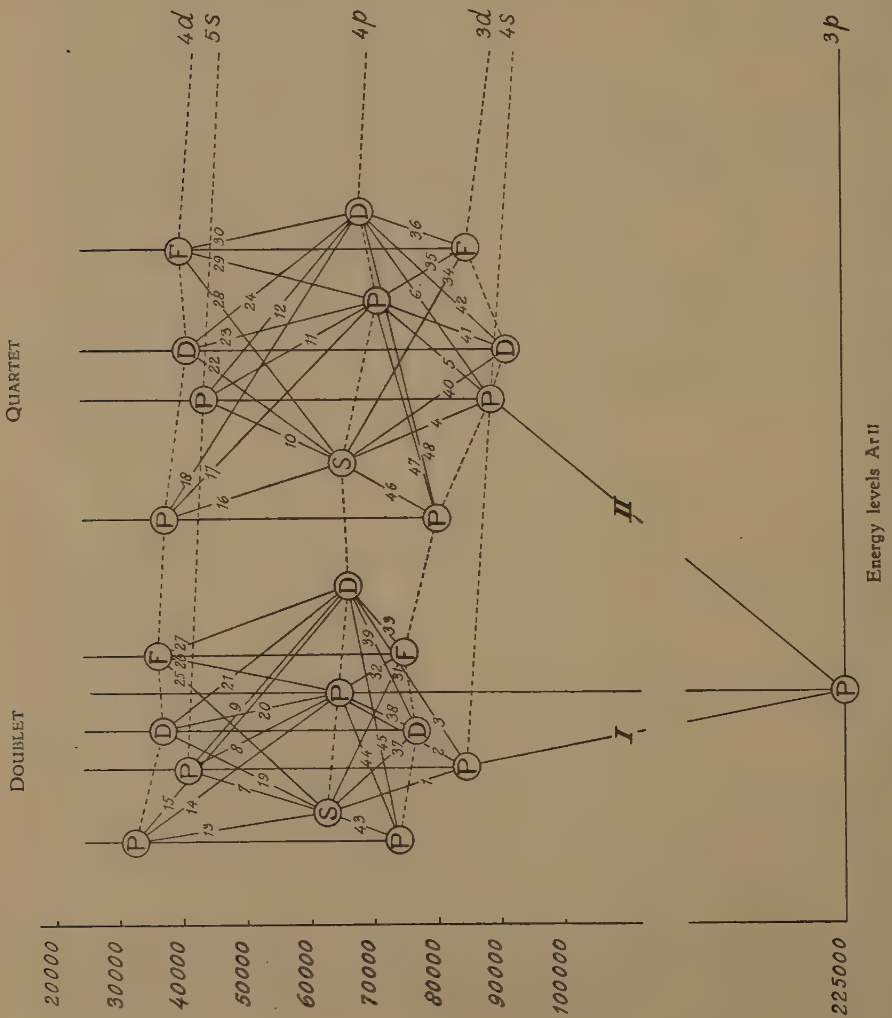
1,51 0,92 (**0**,28) (**0**,28) 0,99 **1,54**.

The g-values are here calculated in the same way to:

$$g\,4s\,^2P_1 = 0{,}67 \text{ and } g\,4p\,^2P_2 = 1{,}23.$$

According to the g-formula one should expect 0,67 and 1,33.

We now can predict with the found g-values the deviating ZEEMAN-



effects of both the other lines of the multiplet. We expect e.g. with the given g-values that the line $\lambda\, 4545{,}040 = 4s\, {}^2P_2 - 4p\, {}^2P_2$ shows the ZEEMAN-effect

$$\pm\, (0{,}05)\ \ (0{,}15)\ \ 1{,}18\ \ 1{,}28\ \ 1{,}38.$$

Observed is

$$\pm \qquad (0{,}16) \qquad 1{,}28.$$

When the coupling is the normal one, that means when we may use the g-formula, of LANDÉ one should expect that the ZEEMAN-effect of this line would show a sharp triplet (0,00) 1,33.

The deviating g-values in the following tables [1]) are calculated in the same way as in this example.

4. *The* ZEEMAN-*effect of the Ar II lines.*

MULTIPLETS 5, 6 and 4.

	4s 4p	$^4P_1$	515.70	$^4P_2$	844.40	$^4P_3$
5.	$^4P_1$	3. 4972.16 20106.4	515.85	6. 4847.783 20622.25		—
	357.30					
	$^4P_2$	5. 5062.019 19749.47	515.60	5. 4933.226 20265.07	844.42	7. 4735.855 21109.49
	307.78					
	$^4P_3$	—		6. 5009.246 19957.53	844.03	9. 4805.993 20801.56
6.	$^4D_1$	8. 4379.657 22826.44	515.72	5. 4282.894 23342.16		—
	260.31					
	$^4D_2$	7. 4430.185 22566.11	515.74	7. 4331.194 23081.85	844.36	3. 4178.344 23926.21
	494.40					
	$^4D_3$	—		9. 4425.995 22587.23	844.38	6. 4266.524 23431.71
	439.40					
	$^4D_4$	—		—		10. 4348.062 22992.30
4.	$^4S_2$	6. 3928.599 25447.20	515.68	9. 3850.565 25962.88	844.21	10. 3729.300 26807.09

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[1]) See for the numbers of the multiplets the figure of the energy levels.

	λ	Termcomb. x—y		ZEEMAN-effect	g_x		g_y		Remarks
					theor.	obs.	theor.	obs.	
5.	5062.019	4s $^4P_1$ — 4p $^4P_2$	L[1] obs.	**(0.47)** **1.27** 2.20	2.67		1.73		1)
	5009.246	4s $^4P_2$ — 4p $^4P_3$	L obs.	**(0.07)** (0.20) **1.40** 1.53 1.67 1.80	1.73		1.60		2)
	4972.16	4s $^4P_1$ — 4p $^4P_1$	L obs.	(0.00) 2.67	2.67		2.67		3)
	4933.226	4s $^4P_2$ — 4p $^4P_2$	L obs.	**(0.00)** 1.73 (0.00) 1.73	1.73	1.73	1.73	1.73	4)
	4847.783	4s $^4P_2$ — 4p $^4P_1$	L obs.	**(0.47)** **1.27** 2.20 **(0.46)** **1.27**	1.73	1.73	2.67	2.67	5)
	4805.993	4s $^4P_3$ — 4p $^4P_3$	L obs.	(0.00) 1.60 (0.00) 1.60	1.60	1.60	1.60	1.60	6)
	4735.885	4s $^4P_3$ — 4p $^4P_2$	L obs.	**(0.07)** (0.20) **1.40** 1.53 1.67 1.80 **(0.00)** **1.48**	1.60	1.60	1.73	1.73	7)
6.	4430.185	4s $^4P_1$ — 4p $^4D_2$	L obs.	**0.47** **(0.73)** 1.93 **0.46** **(0.74)** 1.93	2.67	2.67	1.20	1.20	8)
	4425.995	4s $^4P_2$ — 4p $^4D_3$	L obs. calc.	**(0.18)** (0.54) **0.83** 1.19 1.55 1.91 **(0.20)** **0.67** 1.13 1.52 — **(0.20)** (0.60) **0.73** 1.13 1.53 1.93	1.73	1.73	1.37	1.33	9)
	4379.675	4s $^4P_1$ — 4p $^4D_1$	L obs.	(1.33) 1.33 (1.33) 1.33	2.67	2.67	0.00	0.00	10)
	4348.062	4s $^4P_3$ — 4p $^4D_4$	L obs.	**(0.09)** (0.26) (0.43) **1.00** 1.17 1.34 1.51 1.69 1.86 **(0.00)** **1.09**	1.60	1.60	1.43	1.43	11)
	4331.194	4s $^4P_2$ — 4p $^4D_2$	L obs.	(0.27) **(0.80)** 0.93 **1.47** 2.00 (0.28) **0.83** **1.47** 2.00	1.73	1.73	1.20	1.20	12)
	4282.894	4s $^4P_2$ — 4p $^4D_1$	L obs.	**(0.87)** 0.87 **2.60** **(0.86)** 0.86 **2.58**	1.73	1.73	0.00	0.00	13)
	4266.524	4s $^4P_3$ — 4p $^4D_3$	L obs. calc.	(0.11) (0.34) **(0.57)** 1.03 1.26 **1.49** 1.71 1.94 (0.66) **1.47** (0.13^5) (0.40^5) **(0.67^5)** 0.92^5 1.19^5 **1.46^5** 1.73^5 2.00^5	1.60	1.60	1.37	1.33	14)
4.	3928.599	4s $^4P_1$ — 4p $^4S_2$	L obs.	**(0.33)** **1.66** 2.33 **(0.33)** **1.66** 2.32	2.67	2.67	2.00	2.00	15)
	3850.565	4s $^4P_2$ — 4p $^4S_2$	L obs.	(0.13) **(0.40)** 1.60 **1.87** 2.13 **(0.38^5)**	1.73	1.73	2.00	2.00	16)
	3729.300	4s $^4P_3$ — 4p $^4S_2$	L obs.	**(0.20)** (0.60) **1.00** 1.40 1.80 **2.20** **(0.20)** (0.60) **1.00** 1.40 1.80 **2.20**	1.60	1.60	2.00	2.00	17

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[1] L = ZEEMAN-effect calculated with LANDÉ's g-formula.
obs. = observed ZEEMAN-effect.
calc. = ZEEMAN-effect calculated with anomalous g-values.

REMARKS.

1) 2) 3) These 3 lines 5062.019, 5009.246 and 4972.16 fall in a region, that is not yet investigated. The qualitative results of LÜTTIG agree with the classification.

4) Sharp triplet.

5) Weak quartet.

6) Sharp triplet.

7) Diffuse triplet.

8) Sharp sextet. See photogram 8.

9) Component 0.67 enlarged. See photogram 16.

10) Sharp doublet. See photogram 10.

11) "Pseudotriplet" Central component enlarged; in both the other large components decrease of intensity to the outside. See photogram 12.

12) Partial coincidence with 4332.035 ($3d\,^4D_2$—$4p\,^4P_1$).

13) Weak, but sharp quartet.

14) Outside components diffuse.

15) Sharp sextet. See photogram 15.

16) This line falls in the *CN* band 3883, therefore the components could not be measured exactly.

17) Completely resolved in 12 components. See photogram 9.

MULTIPLETS 3, 2, 1.

	4s \ 4p	$^2P_2$	1014.74	$^2P_1$
3.	$^2D_3$	8. 4879.824 20486.83		—
	663.02			
	$^2D_2$	6. 4726.847 21149.85	1014.71	6. 4965.059 20135.14
2.	$^2P_1$	6. 4657.889 21462.96	1014.86	3. 4889.06 20448.1
	532.90			
	$^2P_2$	7. 4545.040 21995.86	1014.60	5. 4764.848 20981.18
1.	$^2S_1$	5. 4375.942 22845.82	1014.74	6. 4579.347 21831.08

	λ	Termcomb. $x—y$		ZEEMAN-effect						g_x		g_y		Remarks
										theor.	obs.	theor.	obs.	
3.	4965.059	$4s\ ^2P_1 — 4p\ ^2D_2$	L	(0.07)	0.73	**0.87**				0.67	0.67	0.80	0.90	1)
			obs.	**(0.00)**		**1.02**								
			calc.	**(0.11^5)**	0.78^5	**1.01^5**								
	4879.824	$4s\ ^2P_2 — 4p\ ^2D_3$	L	**(0.07)**	(0.20)	**1.00**	1.13	1.27	1.40	1.33	1.33	1.20	1.24	2)
			obs.	**(0.00)**		**1.14**								
			calc.	**(0.05)**	(0.13^5)	**1.10^5**	1.19^5	1.28^5	1.37^5					
	4726.847	$4s\ ^2P_2 — 4p\ ^2D_2$	L	(0.27)	0.53	(0.80)	**1.07**	1.60		1.33	1.33	0.80	0.90	3)
			obs.	(0.24)	**(0.65)**		**1.12**	1.54						
			calc.	(0.22)	**(0.65)**	0.69	**1.12**	1.54						
2.	4889.06	$4s\ ^2P_1 — 4p\ ^2P_1$	L	(0.00)	0.67					0.67	0.67	0.67	0.99	4)
			obs.	(0.17)	0.83									
			calc.	(0.16)	0.83									
	4764.848	$4s\ ^2P_1 — 4p\ ^2P_2$	L	**1.67**	1.00	**(0.33)**	**(0.33)**	1.00	**1.67**	0.67	0.67	1.33	1.23	5)
			obs.	**1.51**	0.92	(0.28)	(0.28)	0.99	**1.54**					
			calc.	**1.51**	0.95	**(0.28)**	**(0.28)**	0.95	**1.51**					
	4657.889	$4s\ ^2P_2 — 4p\ ^2P_1$	L	**1.67**	1.00	**(0.33)**	**(0.33)**	1.00	**1.67**	1.33	1.33	0.67	0.99	6)
			obs.	**1.50^5**	1.14^5	**(0.16^5)**	**(0.16^5)**	1.21	**1.51**					
			calc.	**1.50**	1.16	**(0.17)**	**(0.17)**	1.16	**1.50**					
	4545.040	$4s\ ^2P_2 — 4p\ ^2P_2$	L	(0.00)	1.33					1.33	1.33	1.33	1.23	7)
			obs.		**(0.16)**		**1.28**							
			calc.	(0.05)	**(0.15)**	1.18	**1.28**	1.38						
1.	4579.347	$4s\ ^2P_1 — 4p\ ^2S_1$	L	(0.67)	1.33					0.67	0.67	2.00	1.68	8)
			obs.	(0.50^5)	1.17									
			calc.	(0.50^5)	1.17^5									
	4375.942	$4s\ ^2P_2 — 4p\ ^2S_1$	L	1.67	**1.00**	(0.33)	(0.33)	**1.00**	1.67	1.33	1.33	2.00	1.68	9)
			obs.	1.51	**1.15**	(0.17)	(0.17)	**1.12**	1.51					
			calc.	1.51	**1.15**	(0.17)	(0.17)	**1.15**	1.51					

REMARKS.

1) Weak triplet. Central component enlarged.

2) Diffuse triplet.

3) Sharp octet. See photogram 2.

4) Weak quartet.

5) Sharp sextet. See photogram 1.

6) Sharp sextet. See photogram 3. (The slight deviations of these values from those given in the original dutch paper are due to recent measurements on the photogram.)

7) See photogram 6.

8) See photogram 5.

9) Component 1.51 weak.

MULTIPLETS 11, 12 and 10.

	5s / 4p	$^4P_1$	729.50	$^4P_2$	627.90	$^4P_3$
11.	$^4P_1$	1. 3669.550 27243.56	729.87	4. 3770.569 26513.69		—
	357.30					
	$^4P_2$	4. 3622.204 27599.69	728.96	3. 3720.467 26870.73	627.99	5. 3809.499 26242.74
	307.78					
	$^4P_3$	—		3. 3678.328 27178.54	627.84	6. 3765.313 26550.70
12.	$^4D_1$	6. 4076.704 24522.72	730.96	3. 4201.946 23791.76		—
	260.31					
	$^4D_2$	5. 4033.872 24783.10	729.04	6. 4156.135 24054.06	627.66	3. 4267.49 23426.4
	494.40					
	$^4D_3$	—		4. 4072.429 24548.48	627.91	4. 4179.329 23920.57
	439.40					
	$^4D_4$	—		—		10. 4103.957 24359.88
10.	$^4S_2$	1. 4564.50 21902.1	729.0	3. 4721.66 21173.1	628.1	2. 4866.00 20545.0

	λ	Termcomb. $x - y$	ZEEMAN-effect		g_x		g_y		Remarks
					theor.	obs.	theor.	obs.	
12.	4156.135	4p $^4D_2$ — 5s $^4P_2$	L obs. calc.	(0.27) **(0.80)** 0.93 **1.47** 2.00 (0.63) (0.21^5) **(0.64)** 0.98^5 **1.41^5** 1.84	1.20	1.20	1.73	1.63	1)
	4076.704	4p $^4D_1$ — 5s $^4P_1$	L obs. calc.	(1.33) 1.33 (1.24^5) 1.24^5 (1.26^5) 1.26^5	0.00	0.00	2.67	2.53	2)
	4033.872	4p $^4D_2$ — 5s $^4P_1$	L obs. calc.	**0.47** (0.73) 1.93 **0.61** **0.53^5** **(0.66^5)** 1.86^5	1.20	1.20	2.67	2.53	3)

	λ	Termcomb. $x - y$		ZEEMAN-effect	g_x theor.	g_x obs.	g_y theor.	g_y obs.	Remarks
11.	3809.499	4p $^4P_2$ — 5s $^4P_3$	L	**(0.07)** (0.20) **1.40** 1.53 1.67 1.80	1.73	1.73	1.60	1.60	4)
			obs.	**(0.00)** **1.47**					
	3770.569	4p $^4P_1$ — 5s $^4P_2$	L	**(0.47)** **1.27** 2.20	2.67	2.67	1.73	1.63	5)
			obs.	**(0.51^5)** **1.09**					
			calc.	**(0.51^5)** **1.11** 2.14^5					
	3765.313	4p $^4P_3$ — 5s $^4P_3$	L	(0.00) 1.60	1.60	1.60	1.60	1.60	6)
			obs.	(0.00) 1.60					
	3720.467	4p $^4P_2$ — 5s $^4P_2$	L	(0.00) 1.73	1.73	1.73	1.73	1.63	7)
			obs.	**0.16^5** **1.70**					
			calc.	(0.05) **(0.15^5)** 1.58 **1.68** 1.78^5					
	3678.328	4p $^4P_3$ — 5s $^4P_2$	L	**(0.07)** (0.20) **1.40** 1.53 1.67 1.80	1.60	1.60	1.73	1.63	8)
			obs.	**(0.00)** **1.57**					
			calc.	**(0.01^5)** (0.04^5) **1.55^5** 1.58^5 1.61^5 1.64^5					
	3622.204	4p $^4P_2$ — 5s $^4P_1$	L	**(0.47)** **1.27** 2.20	1.73	1.73	2.67	2.53	9)
			obs.	**(0.42)** **1.34**					
			calc.	(0.40) **1.33^5** 2.12^5					

REMARKS.

1) Diffuse, weak doublet.
2) Coincidence with 4077.057 (4p $^2D_2$—5s $^2P_1$). Only one component can be measured in regard to the *Fe*-line 4076.642, therefore the measurement is not very exact.
3) Diffuse doublet.
4) "Pseudotriplet", diffuse.
5) Sharp quartet, component 0.51 exactly measurable.
6) Sharp triplet.
7) Weak, very diffuse quartet.
8) "Pseudotriplet", diffuse.
9) Very weak quartet.

MULTIPLETS 9, 8 and 7.

	4p \ 5s	$^2P_2$	824.00	$^2P_1$
9.	$^2D_3$	10. 4103.957 24359.88		—
	663.02			
	$^2D_2$	5. 4218.683 23697.43	823.16	2. 4077.057 24520.59
8.	$^2P_1$	4. 4275.167 23384.36	823.9	1. 4129.67 24208.2
	532.90			
	$^2P_2$	2. 4375.031 22850.68	824.32	6. 4222.679 23675.00
7.	$^2S_1$	(4543.68) (22002.44)	824.00	8. 4379.657 22826.44

λ	Termcomb. x—y	ZEEMAN-effect							g_x		g_y		Remarks
									theor.	obs.	theor.	obs.	
4218.683	4p $^2D_2$ — 5s $^2P_2$	L	(0.27)	0.53	(0.80)	1.07	1.60		0.80	0.90	1.33	1.43	1)
		obs.			(0.79)								
		calc.	(0.26^5)	0.63^5	(0.79^5)	1.16^5	1.69^5						
4103.957	4p $^2D_3$ — 5s $^2P_2$	L	(0.07)	(0.20)	1.00	1.13	1.27	1.40	1.20	1.24	1.33	1.43	2)
		obs.	(0.00)		0.97								
		calc.	(0.09^5)	(0.28^5)	0.95^5	1.14^5	1.33^5	1.52^5					
4077.057	4p $^2D_2$ — 5s $^2P_1$	L	(0.07)	0.73	0.87				0.80	0.90	0.67	0.81	3)
		obs.	(0.00)		0.94								
		calc	(0.04^5)	0.85^5	0.94^5								
4275.167	4p $^2P_1$ — 5s $^2P_2$	L	(0.33)	1.00	1.67				0.67	0.99	1.33	1.43	4)
		obs.	(0.22)		1.64^5								
		calc.	(0.22)	1.21	1.65								
4222.679	4p $^2P_2$ — 5s $^2P_1$	L	(0 33)	1.00	1.67				1.33	1.23	0.67	0.81	5)
		obs.	(0.21)	1.03	1.45								
		calc.	(0.21)	1.02	1.44^5								
4379.657	4p $^2S_1$ — 5s $^2P_1$	L	(0.67)	1.33					2.00	1.68	0.67	0.81	6)
		obs.											
		calc.	(0.43^5)	1.24^5									

REMARKS.

1) Diffuse doublet, decrease of intensity clearly to the inside.
2) Coincidence with 4103.957 ($4p\,^4D_4$—$5s\,^4P_3$). "Pseudotriplet".
3) Partial coincidence with 4076.704 ($4p\,^4D_1$—$5s\,^4P_1$).
4) Sharp quartet. (The slight deviations of these values from those given in the original dutch paper are due to recent measurements on the photogram.)
5) Sharp sextet.
6) Coincidence with the strong doublet 4379.657 ($4s\,^4P_1$—$4p\,^4D_1$). We measured a distance 1.65, which perhaps can agree with the distance from —0.43^5 to +1.24^5 = 1.68 of the calculated ZEEMAN-effect. See photogram 10.

MULTIPLETS 41, 42 and 40.

	3d 4p	$^4D_1$	107.00	$^4D_2$	149.53	$^4D_3$	153.90	$^4D_4$
41.	$^4P_1$	5. 4352.198 22970.46	106.91	5. 4332.035 23077.37		—		—
	357.30							
	$^4P_2$	4. 4420.943 22613.14	107.25	6. 4400.101 22720.39	149.61	5. 4371.334 22869.90		—
	307.78							
	$^4P_3$	—		4. 4460.512 22412.69	149.26	5. 4431.002 22561.95	153.86	7. 4400.986 22715.81

MULTIPLETS 41, 42 and 40 (*Continued*).

	3d / 4p	$^4D_1$	107.00	$^4D_2$	149.53	$^4D_3$	153.90	$^4D_4$
42.	$^4D_1$	3. 3891.400 25690.44	107.05	4. 3875.256 25797.49		—		—
	260.31							
	$^4D_2$	3. 3931.232 25430.14	106.87	4. 3914.781 25537.01	149.61	3. 3891.978 25686.62		—
	494.40							
	$^4D_3$	—		4. 3992.046 25042.75	149.57	5. 3968.346 25192.32	153.84	4. 3944.259 25346.16
	439.47							
	$^4D_4$	—		—		4. 4038.816 24752.76	153.95	8. 4013.852 24906.71
40.	$^4S_2$	(3551.18) (28311.05)	106.66	1. 3517.929 28417.71	148.31	1. 3499.665 28566.02		—

	λ	Termcomb. x—y		ZEEMAN-effect	g_x theor.	g_x obs.	g_y theor.	g_y obs.	Remarks
41.	4431.002	3d $^4D_3$—4p $^4P_3$	L obs. calc.	(0.11) (0.34) **(0.57)** 1.03 1.26 **1.49** 1.71 1.94 **(0.53)** (0.11) (0.33) **(0.55)** 1.05 1.27 **1.49** 1.71 1.93	1.37	1.38	1.60	1.60	1)
	4400.986	3d $^4D_4$—4p $^4P_3$	L obs.	**(0.09)** (0.26) (0.43) **1.00** 1.17 1.34 1.51 1.69 1.86 **(0.00)** **1.08**	1.43	1.43	1.60	1.60	2)
	4400.101	3d $^4D_2$—4p $^4P_2$	L obs.	(0.27) **(0.80)** 0.93 **1.47** 2.00 **0.83** **1.49**	1.20	1.20	1.73	1.73	3)
	4371.334	3d $^4D_3$—4p $^4P_2$	L obs. calc.	**(0.18)** (0.54) **0.83** 1.19 1.55 1.91 **(0.18)** (0.55) **0.86** **(0.18)** (0.57) **0.85** 1.20 1.56 1.91	1.37	1.38	1.73	1.73	4)
	4352.198	3d $^4D_1$—4p $^4P_1$	L obs.	(1.33) 1.33 (1.32) 1.32	0.00	0.00	2.67	2.67	5)
	4332.035	3d $^4D_2$—4p $^4P_1$	L obs.	**0.47** **(0.73)** 1.93 **0.47**	1.20	1.20	2.67	2.67	6)
42.	4013.852	3d $^4D_4$—4p $^4D_4$	L obs.	(0.00) 1.43 (0.00) 1.44	1.43	1.43	1.43	1.43	7)
	3968.346	3d $^4D_3$—4p $^4D_3$	L obs. calc.	(0.00) 1.37 **(0.00)** **1.36** (0.02^5) (0.07^5) **(0.12^5)** 1.12^5 1.25^5 1.30^5 **1.35^5** 1.40^5 1.45^5	1.37	1.38	1.37	1.33	8)
	3944.259	3d $^4D_4$—4p $^4D_3$	L obs. calc.	**(0.02)** (0.09) (0.14) 1.29 1.34 1.40 1.46 1.51 **1.57** **1.47** **(0.00)** **1.57** **(0.04^5)** (0.14^5) (0.24^5) 1.18^5 1.28^5 1.37^5 1.47^5 1.57^5 **1.67^5**	1.43	1.43	1.37	1.33	9)

REMARKS.

1) Partial coincidence with 4331.194 ((*s* $^4P_2$—*4p* $^4D_1$). The 0.53 components are weak. With the micro-projection apparatus 2 very weak components can yet be observed outside the 0.53 components.
2) "Pseudotriplet". Central component enlarged, in both the other large components decrease of intensity clearly to the outside.
3) Weak, partial coincidence with 4400.986 (*3d* $^4D_4$—*4p* $^4P_3$).
4) Partial coincidence with 4370.785.
5) Sharp doublet. See photogram 11.
6) Partial coincidence with 4331.194 (*4s* $^4P_2$—*4p* $^4D_1$), therefore not exactly measurable.
7) Sharp triplet.
8) All components diffuse.
9) Very asymmetric triplet, diffuse.

The following lines, combinations with the higher *4d*-terms, generally are weak lines of the spectrum. Therefore the separations of these lines are not resolved so finely, that all *g*-values of these levels could be calculated.

MULTIPLETS 23, 24, 30, 18 and 16.

	λ	Termcomb. *x—y*	ZEEMAN-effect		Remarks
24.	3826.826	4p $^4D_3$ — 4d $^4D_3$	L. obs.	(0.00) 1.37 (0.00) 1.44	1)
	3780.868	4p $^4D_4$ — 4d $^4D_4$	L. obs.	(0.00) 1.43 (0.00) 1.41	
	3763.565	4p $^4D_4$ — 4d $^4D_3$	L. obs.	(0.02) (0.09) (0.14) 1.29 1.34 1.40 1.46 1.51 1.57 (0.00) 1.51	2)
23.	3535.364	4p $^4P_1$ — 4d $^4D_2$	L. obs.	0.47 (0.73) 1.93 0.60	3)
	3514.426	4p $^4P_2$ — 4d $^4D_3$	L. obs.	(0.18) (0.54) 0.83 1.19 1.55 1.91	4)
	3509.811	4p $^4P_1$ — 4d $^4D_1$	L. obs.	(1.33) 1.33 1.32^5	5)
	3491.573	4p $^4P_3$ — 4d $^4D_4$	L. obs.	(0.09) (0.26) (0.43) 1.00 1.17 1.34 1.51 1.69 1.86 (0.00) 1.09	6)
	3491.290	4p $^4P_2$ — 4d $^4D_2$	L. obs.	(0.27) (0.80) 0.93 1.47 2.00 (0.72^5) 1.49^5	7)
	3476.776	4p $^4P_3$ — 4d $^4D_3$	L. obs.	(0.11) (0.34) (0.57) 1.03 1.26 1.49 1.71 1.94 1.46 (0.53^5) (0.53^5) 1.58	8)
	3466.3_{83}	4p $^4P_2$ — 4d $^4D_1$	L. obs.	(0.87) 0.87 2.60 2.33 (0.74) (0.74) 2.43	9)

REMARKS.

1) Both components of the doublet enlarged, and decrease of intensity clearly to the outside. Calculation with $g4s\,^4P_2 = 2.67$ and $g4p\,^2D_3 = 1.24$.
2) Weak triplet. Calculation with $g4s\,^4P_1 = 2.67$ and $g4p\,^2D_2 = 0.90$.
3) Coincidence with 3974.496 ($4p\,^2P_1$—$4d\,^2D_2$). Calculation with $g4s\,^4P_2 = 1.73$ and $g4p\,^2P_2 = 1.23$.
4) Very weak doublet. Calculation with $g4p\,^4P_1 = 2.67$ and $g5s\,^2P_2 = 1.43$.

UNCLASSIFIED LINES.

λ	ZEEMAN-effect		Remarks	λ	ZEEMAN-effect		Remarks
4609.634	obs.	(0.00) 1.06	1)	4042.889	obs.	(0.00) 0.80	15)
4598.86	obs.	0.77^5	2)	3994.885	obs.	(0.61) 1.37	16)
4589.901	obs.	(0.00) 0.90	3)	3803.231	obs.	(0.00) 1.17	17)
4481.833	obs.	(0.00) 1.19	4)	3766.136	obs.	(0.00) 1.32	18)
4474.998	obs.	(0.00) 0.84	5)	3753.572	obs.	(0.30) 0.94^5	19)
4448.953	obs.	1.05 (0.00) 1.32	6)	3724.547	obs.	0.82^5	20)
4439.369	obs.	(0.00) 0.83	7)	3718.253	obs.	(0.00) 0.91	21)
4433.867	obs.	(0.00) 1.06	8)	3660.485	obs.	1.29 (0.00) 1.21	22)
4385.07	obs.	(0.38) 0.87	9)	3639.872	obs.	(0.00) 1.05	
4370.758	obs.	(0.00) 0.80		3561.063	obs.	(0.00) 1.04	23)
4367.782	obs.	(0.54) 1.49	10)	3545.855	obs.	(0.00) 0.97	24)
4277.558	obs.	(0.00) 1.09	11)	3478.260	obs.	(0.00) 1.71	
4237.235	obs.	1.53^5 (0.77^5) (0.77^5) 1.60	12)	3376.468	obs.	(0.00) 1.12	
4131.763	obs.	(0.00) 0.77	13)	3350.972	obs.	(0.00) 0.80	
4072.009	obs.	(0.00) 1.18					
4052.961	obs.	(0.33) 1.00 1.67	14)				

REMARKS.

1) Sharp triplet. See photogram 4.
2) Diffuse doublet.
3) Rather sharp triplet.
4) Sharp triplet.
5) Sharp triplet.
6) Asymmetric triplet. See photogram 7.
7) Sharp triplet.
8) Rather diffuse triplet.
9) Rather sharp quartet Calculation with $g = 2$ ($j = 1$) and $g = 1.25$ ($j = 2$) gives (0.37^5) 0.87^5. Perhaps combination with $^2S_1$.

10) Calculation with $g=2$ $(j=1)$ and $g=0.94$ $(j=1)$ gives (0.53) 1.47.
11) Diffuse strong triplet.
12) Diffuse quartet.
13) Diffuse triplet.
14) Calculation with $g\,{}^2S_1 = 2.00$ and $g\,{}^2P_2 = 1.33$ gives (0.33) 1.00 1.67 sharp sextet. Perhaps terms of the configuration upon basicterm ${}^1D$ or ${}^1S$. See photogram 13.
15) Sharp triplet.
16) Sharp quartet. Calculation with $g\,{}^2S_1 = 2.00$ and $g\,{}^2P_1 = 0.76$ gives (0.62) 1.38. Perhaps terms of the configuration upon basicterm ${}^1D$ or ${}^1S$.
17) Rather sharp triplet.
18) Coincidence with 3765.313 ($4p\,{}^4P_3$—$5s\,{}^4P_3$).
19) Quartet. Calculation with $g=0.65$ $(j=1)$ and $g=1.25$ $(j=1)$ gives (0.30) 0.95.
20) Diffuse doublet.
21) Strong, rather sharp triplet.
22) Diffuse asymmetric triplet.
23) Strong triplet.
24) Strong triplet. Coincidence with 3545.642.

5. *g-sums.*

In § 3 we showed that anomalous *g*-values can occur. According to theoretical considerations of Heisenberg 1), Pauli 2) and Landé 3) one expects however, although the *g*-values of the individual terms deviate, that the *g*-sum of certain term groups will remain constant. The experimental data are not yet extensive; we refer to the notes of Back 4), Green and Loring 5), Shenstone 6). From the terms which arise from the coupling of one electron, one must join to groups the terms with equal *j*, then the sum of the *g*-values of the terms of such a group is constant and can be calculated beforehand. This sum is equal to the sum of Landé's *g*-values for these terms.

In the following tables the theoretical *g*-values (g_L) and the observed *g*-values (g_{obs}) are given for the terms arising from the coupling of a 4*s*, 4*p* and 5*s* electron. The last column shows the *g*-sums obtained from Landé's values and the observed *g*-sums.

In the case of the *3d-terms* the *g*-sum rule cannot be checked because all terms are not yet found.

The terms of the 4*s*-electron all have normal *g*-values and the *g*-sum rule is of course fulfilled.

Among the 13 terms arising from the coupling of the *4p-electron* there are 8 quartet terms of which 7 have normal *g*-values, while the 5 doublet

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1) Heisenberg, Zeitschr. f. Phys., **8**, 273, 1922.
2) Pauli, Zeitschr. f. Phys., **16**, 155, 1923.
3) Landé, Zeitschr. f. Phys., **19**, 112, 1923; Ann. der. Phys., **76**, 273, 1925.
4) Back, Ann. der Phys., **76**, 317, 1925.
5) Green and Loring, Phys. Rev., **30**, 574, 1927.
6) Shenstone, Phys. Rev., **30**, 255, 1927.

TERMS OF THE 4s-ELECTRON.

		$^4P_3$	$^4P_2$	$^4P_1$	$^2P_2$	$^2P_1$	
j = 1	g_L			2.67		0.67	$\Sigma g_L = 3.34$
	$g_{obs.}$			2.67		0.67	$\Sigma g_{obs.} = 3.34$
j = 2	g_L		1.73		1.33		$\Sigma g_L = 3.06$
	$g_{obs.}$		1.73		1.33		$\Sigma g_{obs.} = 3.06$
j = 3	g_L	1.60					$\Sigma g_L = 1.60$
	$g_{obs.}$	1.60					$\Sigma g_{obs.} = 1.60$

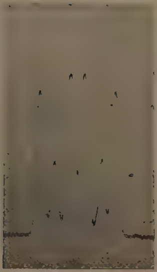
TERMS OF THE 5s-ELECTRON

		$^4P_3$	$^4P_2$	$^4P_1$	$^2P_2$	$^2P_1$	
j = 1	g_L			2.67		0.67	$\Sigma g_L = 3.34$
	$g_{obs.}$			2.53		0.81	$\Sigma g_{obs.} = 3.34$
j = 2	g_L		1.73		1.33		$\Sigma g_L = 3.06$
	$g_{obs.}$		1.63		1.43		$\Sigma g_{obs.} = 3.06$
j = 3	g_L	1.60					$\Sigma g_L = 1.60$
	$g_{obs.}$	1.60					$\Sigma g_{obs.} = 1.60$

TERMS OF THE 4P-ELECTRON.

		$^4P_3$	$^4P_2$	$^4P_1$	$^4D_4$	$^4D_3$	$^4D_2$	$^4D_1$	$^4S_2$	$^2D_3$	$^2D_2$	$^2P_1$	$^2P_2$	$^2S_1$	
j = 1	g_L			2.67				0.00				0.67		2.00	$\Sigma g_L = 5.34$
	$g_{obs.}$			2.67				0.00				0.99		1.68	$\Sigma g_{obs.} = 5.34$
j = 2	g_L		1.73				1.20		2.00		0.80		1.33		$\Sigma g_L = 7.06$
	$g_{obs.}$		1.73				1.20		2.00		0.90		1.23		$\Sigma g_{obs.} = 7.06$
j = 3	g_L	1.60				1.37				1.20					$\Sigma g_L = 4.17$
	$g_{obs.}$	1.60				1.33				1.24					$\Sigma g_{obs.} = 4.17$
j = 4	g_L				1.43										$\Sigma g_L = 1.43$
	$g_{obs.}$				1.43										$\Sigma g_{obs.} = 1.43$

C. J. BAKKER, T. L. DE BRUIN AND P. ZEEMAN: THE ZEEMAN-EFFECT OF THE SPECTRUM OF IONIZED ARGON (AR II).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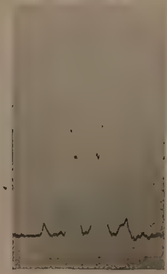
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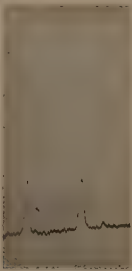
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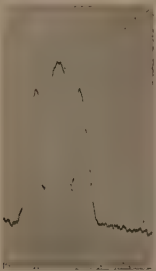
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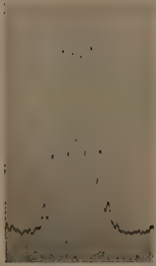
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terms all have anomalous g-values. The agreement of the g-sums is excellent.

The terms of the 5*s-electron* all have anomalous g-values in contradiction with the 4*s*-electron (thus Preston's rule does not hold here).

Some terms of the 4*d-electron* are uncertain, thus it is impossible to check the g-sum rule in this case. The $^4F_5$-term is the only term with $j = 5$. According to the g-sum rule this term must have a normal g-value, which has been found.

The asymmetries of the Zeeman-pattern seem to be stronger when anomalous g-values appear. (See table.)

6. *Summary.*

The Zeeman-effect of 110 *Ar II* lines has been investigated. The analysis of the spectrum given by one of us (de Br.) has been checked by the Zeeman-effect. The Zeeman-effect shows the existence of normal and anomalous couplings in the *Ar II* spectrum; the coupling is anomalous in the higher energy levels. The g-sum rule has been checked for several term groups.

Laboratory „Physica" of the University of
Amsterdam.

June 1928.

The photograms are made with a photometer (made by the firm of Zeiss), provided with photo-electric cell and electrometer.

Photograms.

1.	4764.848	$4s\,^2P_1$—$4p\,^2P_2$	9.	3729.300	$4s\,^4P_3$—$4p\,^4S_2$
2.	4726.846	$4s\,^2P_2$—$4p\,^2D_2$	10.	4379.675	$4s\,^4P_1$—$4p\,^4D_1$
3.	4657.889	$4s\,^2P_2$—$4p\,^2P_1$	11.	4352.198	$3d\,^4D_1$—$4p\,^4P_1$
4.	4609.634		12.	4348.062	$4s\,^4P_3$—$4p\,^4D_4$
5.	4579.347	$4s\,^2P_1$—$4p\,^2S_1$	13.	4052.961	
6.	4545.040	$4s\,^2P_2$—$4p\,^2P_2$	14.	3932.567	$4p\,^4S_2$—$4d\,^4P_2$
7.	4448.953		15.	3928.599	$4s\,^4P_1$—$4p\,^4S_2$
8.	4430.185	$4s\,^4P_1$—$4p\,^4D_2$	16.	4425.995	$4s\,^4P_2$—$4p\,^4D_3$

Physics. — *New measurements about the way in which the dielectric constant of liquid helium depends on the temperature.* By M. WOLFKE and W. H. KEESOM. (Communication N⁰. 192*a* from the Physical Laboratory at Leiden).

(Communicated at the meeting of June 30, 1928)

§ 1. *Introduction.* The first measurements about the way, in which the dielectric constant of liquid helium depends on the temperature, were made by us in the preceding year [1]). These measurements were, it is true, only preliminary, but yet sufficient in order to fix the general course of that quantity. We thereby found, that the dielectric constant of liquid helium shows a jump at about 2.3° K. This observation became for us the starting-point for a further research into other properties of the liquid helium, especially at the temperature mentioned and led us to the discovery of the existence of two different modifications of the liquid helium [2]).

In the course of this year we undertook further measurements in this direction, in order to fix with greater exactness the values of the dielectric constant of the liquid helium at different temperatures.

§ 2. *The apparatus.* This one was nearly the same, as at the former measurements [3]).

The first series of measurements took place on March 15th last. In order to be more sure of the temperature of the liquid in the measuring-condenser, the glassvessel *g* (see Fig. 3 Comm. N⁰. 190*a*) had therefore been taken away, so that the condenser *C* was at once in the cryostat-bath.

Meanwhile during these measurements it was stated, that the temperature of the liquid helium in the cryostat was not sufficiently constant, so that consequently the regularity of the results was disturbed. While in general at exact measurements the helium in the cryostat is being continually stirred by means of special automatic stirrers, we had not put in a stirrer at this first series of measurements because of the difficulties attending the construction of it: in the first place room was wanted in the Dewarvessel, in which the measuring-condenser is placed and further we feared an influence of the moving parts of the stirrer on the capacity of the measuring-condenser. However, these difficulties have been solved by a construction of the stirrer, which has been worked out by Mr. G. J. FLIM. The principal thing of this construction is, that the moving part of the stirrer *R* (Fig. 1) is enclosed sideways in a metal cover, which is joined to the earthed part of the measuring-condenser *C*. A careful examination of this apparatus showed, that a difference in the place of the movable part has no influence on the capacity.

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[1]) These Proc. **31**, 81, 1928. Comm. Leiden N⁰. 190*a*.

[2]) These Proc. **31**, 90, 1928. Comm. Leiden N⁰. 190*b*.

[3]) Compare Comm. Leiden N⁰. 190*a*, Fig. 1.

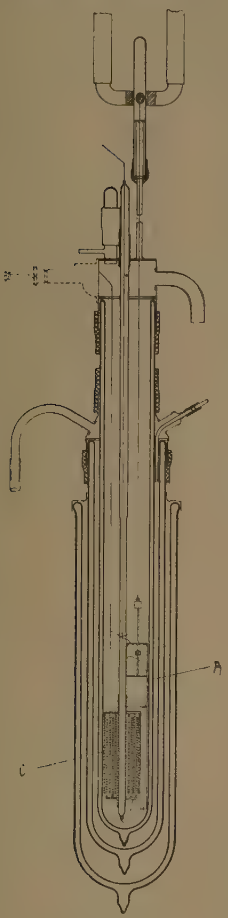

Fig. 1.

A new series of measurements was then made on March 23rd. From these it appeared however, that the results were not yet sufficiently exact; moreover, the readings from the micro-condenser showed a distinct course. On closer research it appeared, that the cause of this inexactitude had to be found in disturbances of the oscillation-condition of the oscillation-circuit 2 [1]), which were caused by the continual changing over of the two condensers C and C_2. In order to eliminate this source of errors, we then have connected the measuring-condenser *C* permanently to the oscillation-circuit. At the same time we have replaced the variable condenser C_1 in the oscillation-circuit 1 by the more accurate normal-condenser C_2, which got off now. After these improvements it appeared, that the sensibility had remained the same, but that the stability of the oscillations had greatly improved, so that during several hours the readings from the micro-condenser belonging to a same measuring-capacity, did not show any important irregularities, nor a distinct course.

The capacity of the measuring-condenser, filled with liquid helium under normal pressure, in the cryostat, was determined in the following way: Instead of the measuring-condenser, a normal-condenser of a definite capacity was put on. After that the capacity of this normal-condenser together with the leads and the micro-condenser was accurately measured. After subtracting the known capacity of the normal-condensor, we obtained the capacity of the leads and the micro-condenser. Then the measuring-condenser filled with helium was put on and the capacity of the latter together with the same leads and the micro-condenser were again accurately measured. From that we then got the capacity of the measuring-condenser filled with helium. It appeared from this that, by putting in the stirrer, the capacity of the measuring-condenser had somewhat increased. For the capacity of the measuring-condenser in liquid helium at the temperature of the normal boiling-point, we found the value : 186.8 cm.

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[1]) Compare Comm. Leiden N⁰. 190a, Fig. 1.

§ 3. *The measurements.* The final measurements took place on March 29th last. They lasted about 5 hours ; during this time the apparatus remained sufficiently constant, so that the readings on the micro-condenser, which took place at the same temperature, but at different times, didn 't show any real differences. In order to eliminate a possible course in the readings, which would be the consequence of a systematic change in the oscillation-condition of the apparatus, the succession of the temperatures, at which we measured the dielectric constant, was chosen in such a way, that always every new point came between two measurements at similar temperature (see table I). The temperature of the helium was deduced in the usual way from the pressure of the helium-bath in virtue of the formula given by KAMERLINGH ONNES and WEBER [1]). These temperatures will possibly suffer a small change, when the results will be fixed of the new measurements [2]) already announced, concerning the saturated vapour-pressure of helium. The adjustments and readings of the micro-condenser took place for each temperature in two series of three times in each case. During the adjustment at the temperature desired and also between both series of three readings, we stirred.

§ 4. *The results.* These are summarized in Table I.

The first column gives the numbers of the points in the order in which they have been measured. In departure from the remaining points (see § 3) for the numbers 10*a* and *b* and 14*a* and *b* the averages of a series of three readings at every time, are given. The meaning of the following columns appears from the readings.

During the evaporation of the helium in the cryostat, the surface of the liquid helium gradually lowered ; at the two last measurements it was already so low, that no sufficient stirring took place any more. Consequently we see, that these last readings on the micro-condenser strongly deviate from those, which were done before, compare for instance N⁰. 18. We therefore shall leave out the last measurements Nos 20 and 21. In connection with this, we already consider N⁰. 19 as doubtful.

From the readings on the micro-condenser, the values of the dielectric constant of the liquid helium can be calculated in the way as given in our former communication [3]), observing the new value of C, being the capacity of the measuring-condenser in the liquid helium at the temperature of its normal boiling-point (§ 2).

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[1]) These Proc. 18, 493. Comm.Leiden N⁰. 147*b*. The readings of the pressure and the calculation of the temperatures have been made by Mr. H. VAN DIJK, phil. cand., whom we heartily thank for this.

[2]) Compare Comm. N⁰. 190*b*, § 3.

[3]) Compare Comm. N⁰. 190*a*, formula (2).

The course of the dielectric constant of the liquid helium with the temperature has been represented in Fig. 2 in virtue of these values.

We see, that the jump in the dielectric constant stated by us in the former measurements, is also seen here. The amount of the jump is, however, considerably smaller than was expected at the first series of measurements

TABLE I.

N^0.	Time	Pressure mm. Hg.	Temp. °K.	Micro-condensator	Dielectric constant
1	13^h 54^m	751.1	4.19	0	1.048 1)
2	14^h 15^m	377.2	3.57	777	1.05179
3	50	387.2	3.59	808	194
4	15^h 3^m	189.6	3.09	1183	377
5	28	385.8	3.58	747	165
6	41	83.6	2.64	1506	535
7	58	188.9	3.09	1218	394
8	16^h 26^m	40.1	2.311	1596	579
9	41	82.3	2.630	1485	525
10*a*	55	38.8	2.296	1628	594
b	58	38.0	2.288	1604	583
11	17^h 4^m	40.4	2.311	1611	586
12	13	42.5	2.335	1584	573
13	25	37.5	2.282	1599	580
14*a*	33	38.65	2.295	1626	593
b	36	38.66	2.295	1611	586
15	43	37.2	2.279	1608	585
16	52	37.9	2.286	1601	581
17	18^h 8^m	20.7	2.055	1535	549
18	23	37.0	2.276	1607	584
19	48	20.0	2.04	1588	575?
[20	19^h 2^m	30.5	2.20	1699	]
[21	14	36.9	2.28	1702	]

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1) This value has been derived from the measurements of M. WOLFKE and H. KAMERLINGH ONNES, Comm. Leiden N^0. 171*b*.

(Comm. N⁰. 190*a*). The course of the dielectric constant near this jump has been given on a scale ten times larger, in the right corner of Fig. 2.

Now we can, in virtue of these new measurements, much more accurately fix the temperature, at which the jump appears [1]). From the measurements

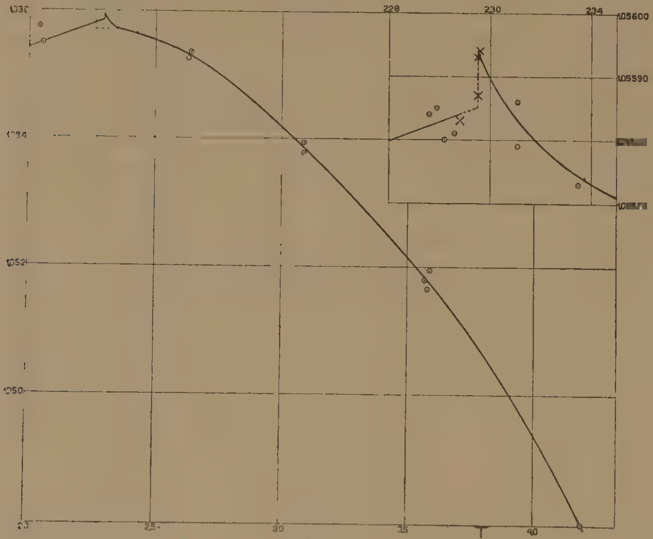

Fig. 2.

N⁰. 10 and 14, we conclude, that the jump appears at a vapour-pressure of 38.65 mm, to which answers a temperature of 2.295° K. [2]) on the accepted temperature scale. So these are (comp. Comm. N⁰. 190*b*) at the same time the pressure, resp. the temperature of the triple-point liquid helium I — liquid helium II — vapour.

§ 5. By deriving values for the density of liquid helium from the measurements of KAMERLINGH ONNES and BOKS [3]) the electric polarization can be calculated according to the formula of CLAUSIUS-MOSOTTI. The results of this calculation are summarized in Table II. In Table II*a* the measurements considered are summarized for that to averages. For the immediate surroundings of the transformation-point, the values for the dielectric constant in Table II*b* have been derived from Fig. 2, the values of the density from Fig. 1 of Comm. N⁰. 190*b*. Especially for 2.295° K., the value of the density of the liquid helium II is still to be considered as preliminary.

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[1]) In the measurements of Comm. N⁰. 190*a* there was a small uncertainty in the temperature, brought about by a small difference in temperature of the liquid helium in the condenser and that of the cryostat bath.

[2]) This last number, after the calculation of the new vapour-pressure measurements (comp. § 3) may consequently still undergo a little change.

[3]) Comm. Leiden N⁰. 170*b*.

From Table II it appears, that the electric polarization, at least in the first approximation, is constant and has the same value for the two modifications of the liquid helium. This makes it probable, that the helium-molecule as such does not undergo any change in internal structure in the transformation-point.

TABLE II*a*.

N⁰.	Pressure mm. Hg.	Temp. °K.	D. C.	ρ	Polarization	
1	751.1	4.19	1.048	0.12566 1)	0.1253	liq. He. I
2, 3, 5	383.4	3.58	1.05179	0.13606	0.1247	„ „ „
4, 7	189.3	3.09	1.05386	0.14152	0.1246	„ „ „
6, 9	82.9	2.63	1.05530	0.14535	0.1245	„ „ „
17	20.7	2.05^5	1.05549	0.14638	0.1241	„ „ II

TABLE II*b*.

		2.35	1.05571	0.14679	0.1242	liq. He. I
		2.295	1.05594	0.14693	0.1246	„ „ „
			1.05585	0.14682	0.1245	„ „ II
		2.26	1.05580	0.14676	0.1242	„ „ „

§ 6. *Comparison with optical data.* As it has appeared from our measurements, that the electric polarization per mass-unity in liquid helium is constant, at least by approximation, it lies at hand to accept, that the same value also holds good for gaseous helium, that is to say, that also at the change of gaseous helium into liquid, the helium-molecule is not altered. In this case we could calculate the dielectric constant of heliumgas from the value obtained by us for the polarization and from the density of the heliumgas. The polarization has been determined by $K—1$; this quantity, however, at the measurements of WOLFKE and KAMERLINGH ONNES 2) has been measured accurately only to 2 %. The absolute values of the polarization consequently, can only lay a claim to this exactness. From this

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1) Corrected according to Comm. N⁰. 172*b*.

2) Comm. Leiden N⁰. 171*b*.

it follows, that the reversed way is better. So we shall calculate from optical data the refractive index for infinitely large wave-length, then we shall determine the polarization with the aid of this value, and after that, we shall calculate the dielectric constant of liquid helium at the temperature of the normal boiling-point.

As well from the measurements of BURTON [1]) as from those of J. KOCH [2]) the value 1.0000346 follows for the refractive index for infinitely large waves. For the density of gaseous helium we take the value measured by BAXTER and STARKWEATHER [3]) $1.7846.10^{-4}$. From this, for the polarization per gramme, according to the formula of LORENZ-LORENTZ follows: 0.1292.

This value is about 4 % larger than the average value from Table II*a*: 0.1246. If, in virtue to this value of the polarization deduced from optical data, we calculate the dielectric constant of liquid helium at the normal boiling-point, then we get 1.0495, instead of the value 1.048 measured by WOLFKE and KAMERLINGH ONNES. The difference comes only to 0.14 %, which means a favourable agreement.

From these considerations we conclude, that the helium-molecule remains the same as well in the gaseous condition, as also in liquid helium I and liquid helium II.

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[1]) W. BURTON, Proc. Roy. Soc. (A) **80**, 390, 1908.

[2]) J. KOCH, Ark. f. Math. Astr. och Fysik **9**, 11, 1913.

[3]) G. P. BAXTER and H. W. STARKWEATHER, Proc. Amer. Acad. **11**, 231, 1925 and **12**, 20, 1926.